The symmetric monoidal category nCob and nTFTs

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1 Definitions

This text is based on a presentation, that was held during a seminar about Homotopical and higher algebra at ETH Zurich. We introduce the category of n-dimensional cobordisms and n-dimensional topological field theories and derive some first properties. These will then be used to completely describe all 1-dimensional topological field theories.

We begin with the definition of the category of n-dimensional cobordisms. For this, we first need to define cobordisms.

Definition 1. Let $n \geq 1$. Let M, M' be two compact, oriented, $n-1$-dimensional manifolds. A cobordism from M to M' is a pair $(B, \imath)$, where B is an oriented, compact, n-dimensional manifold, and $\imath : \partial B \to \bar{M} \coprod M'$ is an orientation-preserving diffeomorphism, where we use the orientation of ∂B induced from the orientation on B and $\bar{M}$ denotes the manifold M with opposite orientation of that given on the oriented manifold M.

We say that two cobordisms $(B, \imath)$ and $(B', \imath')$ are equivalent iff there exist an orientation preserving diffeomorphism $\Phi : B \to B'$, such that $\Phi|_{\partial B} = \imath'^{-1} \circ \imath$.

Note that this defines an equivalence relation on the set of all cobordisms from M to M'.

Remark: When taking the disjoint union of M and M', we need to avoid the problem that the two manifolds might intersect. Therefore, we define $M \coprod M' := (M \times 0) \cup (M' \times 1)$. We will use this convention throughout the whole text.

With this we can define our category.

Definition 2. (Category of (oriented) cobordisms) Let $n \geq 1$. We define the category $nCob$ of n-dimensional cobordisms as follows:

The objects of $nCob$ are the compact, oriented, boundaryless $(n-1)$-dimensional manifolds.

The morphisms $Mor(M, M')$ are the equivalence classes of oriented cobordisms from M to M'.

The composition of two morphisms is defined as follows: Let $[(B,\imath)] \in Mor(M,M')$ and $[(B',\imath')] \in Mor(M',M'')$, where $[\cdot]$ denotes the equivalence class of the given representative. With this, we can define $B \cup_{M'} B' := \frac{B \coprod B'}{\sim}$, where $\sim$ is the equivalence relation, generated by the pairs $\imath^{-1}(p) \sim \imath'^{-1}(p)$ for all $p \in M'$. Moreover, we define

$$\begin{aligned} \tilde{\imath} : \partial(B \cup_{M'} B') &\to \bar{M} \coprod M'' \\ \tilde{\imath}([p]) = \imath(p) \qquad & \text{for } p \in \imath^{-1}(\bar{M}) \\ \tilde{\imath'}([p]) = \imath'(p) \qquad & \text{for } p \in \imath'^{-1}(M'') \end{aligned}$$

The pair $(B \cup_{M'} B', \tilde{\imath})$ is a cobordism from M to M'' and we define its equivalence class to be the composition $[(B,\imath)][(B',\imath')]$.

The identity element in $Mor(M,M)$ is given by $[(B,\imath)]$, where $B = M \times [0,1]$ and $\imath = id$

Now we are left to define the tensor product in our category. We define it to be the map

$$\otimes : nCob \times nCob \to nCob$$

$$(M,N) \mapsto M \coprod N$$

$$([(B,\imath)],[(B',\imath')]) \mapsto [(B \coprod B', \imath \coprod \imath')]$$

where $\imath \coprod \imath'$ denotes the map that is induced naturally by $\imath$ and $\imath'$ on $B \coprod B'$.

The neutral element of this tensor product is the empty manifold $\emptyset$.

Remark: Given this definition, the question arises about its well-definedness. Most of the things are rather easy to check - like associativity of the composition. However, one may wonder, how we can make $B \cup_{M'} B'$ into a manifold such that $\tilde{\imath}$ is an orientation-preserving diffeomorphism. Since taking the quotient only gives us a topological space, we need to choose a differentiable structure on this topological space, which has to be compatible with the differentiable structures on B and B', meaning that the natural inclusion-maps $j : B \to B \cup_{M'} B'$ and $j' : B \cup_{M'} B'$ should be diffeomorphisms onto their images. Not only do we need the existence of such a differentiable structure, but to make our definition above independent of the choice of such a structure we need that for two different structures $\mathcal{A}, \mathcal{B}$ there is an orientation-preserving diffeomorphism from $(B \cup_{M'} B', \mathcal{A})$ to $(B \cup_{M'} B', \mathcal{B})$ which is compatible with the map $\tilde{\imath}$ on the boundary.

This is in fact true, however it involves some work which we do not want to discuss here.

Construction: An orientation-preserving diffeomorphism $h : M \to N$ defines a cobordism from M to N, which is in fact a representative of an isomorphism in the category of cobordisms between M and N. This cobordism is constructed as follows:

Let $B := M \times [0,1]$ and define

$$\begin{aligned} \imath : (M \times 0) \cup (M \times 1) &\to \bar{M} \coprod N \\ (p,0) &\mapsto (p,0) \\ (p,1) &\mapsto (h(p),1) \end{aligned}$$

This defines a cobordism from M to N, based on the orientation-preserving diffeomorphism h.

Remark: The tensor product on a symmetric monoidal category is always equipped with several isomorphisms for associativity and symmetry. These can be constructed, using the construction above and the natural, orientation-preserving diffeomorphism between $(M \coprod N) \coprod P$ and $M \coprod (N \coprod P)$ respectively $M \coprod N$ and $N \coprod M$. We shall denote the isomorphism between $M \coprod N$ and $N \coprod M$ by $\mathrm{swap}_{M,N}$.

After we defined the category of n-dimensional cobordisms, we can now define n-dimensional topological field theories.

Definition 3. (topological field theories) Let $n \geq 1$ and k a field. An n-dimensional topological field theory (TFT) is a symmetric monoidal functor Z from the category $nCob$ to the category $Vect(k)$ of vector spaces over k.

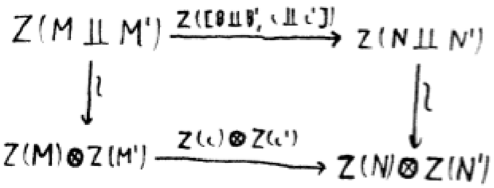
Let us recall what this means: Z sends $n-1$-dimensional, closed manifolds to vector spaces over k and cobordism classes in $Mor(M,N)$ to linear maps from $Z(M)$ to $Z(N)$. Moreover, Z is equipped with isomorphisms which give us the following identifications:

$$Z(\emptyset) \simeq k$$

$$Z(M \coprod N) \simeq Z(M) \otimes Z(N)$$

$$Z(M \times [0,1]) = id_{Z(M)} : Z(M) \to Z(M)$$

and for two cobordisms $c := [(B, \imath)] \in Mor(M,N)$, $c' := [(B', \imath')] \in Mor(M', N')$ the following diagram commutes:



We will denote the set of all n-dimensional TFTs by nTFT.

Remark: Since Z is a symmetric functor, it sends $\text{swap}_{M,N}$ to the canonical isomorphism $\text{swap}_{Z(M),Z(N)} : Z(M) \otimes Z(N) \to Z(N) \otimes Z(M)$ between $Z(M) \otimes Z(N)$ and $Z(N) \otimes Z(M)$, which sends $v \otimes w$ to $w \otimes v$.

In the discussion below, some cobordism classes will play an important role. Therefore we will mention them now and give them a special name.

Let M be an object of $nCob$. Then we have the cobordism class which can be represented by $(M \times [0,1], \imath)$, where $\imath : (M \times 0) \cup (M \times 1) \to \bar{M} \coprod M$ is the identity (note that $\imath$ is only the identity map for technical reasons in the definition of $\coprod$).

This cobordism class can be interpreted in several different ways:

1) As the identity cobordism from M to itself.

2) As the identity cobordism from $\bar{M}$ to itself.

3) As a cobordism from $M \coprod \bar{M}$ to the empty manifold $\emptyset$.

4) As a cobordism from the empty manifold $\emptyset$ to $\bar{M} \coprod M$.

Using the notation in our category, we denote the first two by id_M respectively $id_{\bar{M}}$. The third and fourth are denoted by $ev_M \in Mor(M \coprod \bar{M}, \phi)$ respectively $coev_M \in Mor(\phi, \bar{M} \coprod M)$.

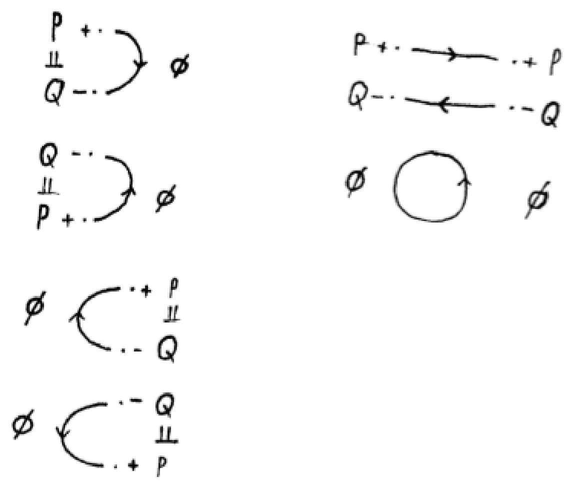
These cobordisms will play an important role in the description of 1TFTs.

2 Description of 1Cob

We will first discuss the oriented case. There, the objects of $1Cob$ are compact, oriented 0-dimensional manifolds without boundaries. These manifolds consist simply of finitely many points, where each point is given a sign which represents its orientation. Let us denote by P the – up to diffeomorphism unique – 0-dimensional manifold, which consists of one point with "positive" orientation, i.e. a +-sign. Define $Q := \bar{P}$. Then every object

$M = \{p_1, \ldots, p_k, q_1, \ldots, q_l\}$ of $1Cob$ consists of k positively oriented and l negatively oriented points where $k, l \geq 0$ and is diffeomorphic to $(\coprod_{i=1}^{k} P) \coprod (\coprod_{j=1}^{l} Q)$ by an orientation-preserving diffeomorphism.

Since two manifolds which are diffemorphic by an orientation-preserving diffeomorphism are isomorphic in cobordism class, we may identify the two objects, when looking at morphism between the objects. So we only look at morphisms between finite disjoint unions of P and Q. Since our cobordisms have to be compact every cobordism is a finite union of connected cobordisms. Therefore, by choosing a representative of our morphism, we can write every morphism as a tensor product of finitely many morphisms whose representatives are all connected. By the classification of compact, 1-dimensional manifolds any representative is diffeomorphic to either S^1 or $[0, 1]$ by an orientation-preserving diffeomorphism. To describe all connected cobordisms – and with this a set of generators of the morphisms of our category – we need to find all possibilities of how these two manifolds can define a cobordism:



This completely describes the oriented category of $1Cob$. In the non-oriented case, we can do just the same, with the following difference: P and Q are diffeomorphic, since we do not care about orientations any longer. The list of cobordisms above is reduced to only five cobordisms which generate all cobordisms between compact 0-dimensional non-oriented manifolds. This completely describes the non-oriented category of $1Cob$.

3 Description of 1TFTs

Again, we will first discuss the oriented case. Following the same lines it is quite easy to see what happens in the non-oriented case. We start with the following

Proposition 4. Let $n \geq 1$ and Z be a $nTFT$. Then for every object M of $nCob$, $Z(M)$ is finite dimensional and the map $Z(ev_M) : Z(M) \otimes Z(\bar{M}) \to k$ is non-degenerate.

Recall that the map $Z(ev_M)$ is non-degenerate if and only if the map

$$\alpha : Z(\bar{M}) \to Z(M)^*$$

$$w \mapsto ev_M(\cdot \otimes w)$$

is an isomorphism of vector spaces, where $Z(M)^*$ denotes the dual space of $Z(M)$.

Also note that we identify $Z(ev_M) : Z(M \coprod \bar{M}) \to Z(\phi)$ with the corresponding map from $Z(M) \otimes Z(\bar{M}) \to k$ by the canonical identifications of these vector spaces.

Proof. We will explicitely construct the inverse of α. For this, we look at the following chain of maps:

$$\beta : Z(M)^* \to Z(M)^* \otimes k \to Z(M)^* \otimes (Z(\bar{M}) \otimes Z(M)) \to Z(\bar{M})$$

$$\omega \quad \longmapsto \quad \omega \otimes 1 \quad \longmapsto \quad \omega \otimes Z(coev_M)(1) \quad \longmapsto \quad \sum_{i=1}^{n} \omega(v_i) w_i$$

where $Z(coev_M)(1) = \sum_{i=1}^{n} w_i \otimes v_i$ for vectors $v_1, \ldots, v_n \in Z(M)$ and $w_1, \ldots, w_n \in Z(\bar{M})$.

We want to show that β is the inverse of α. For this, we consider the identity cobordism id_M for any object M and its decomposition into the two cobordisms $id_M \otimes coev_M$ and $ev_M \otimes id_M$ as shown in the picture below.

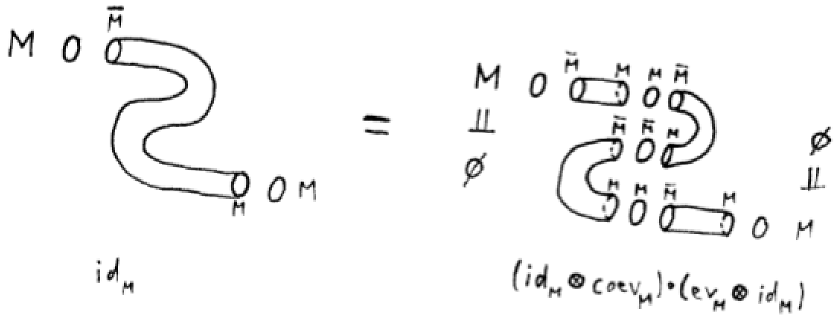
Since the left-hand-side of this picture is the identity, any $nTFT$ Z sends it to the identity map on $Z(M)$. Using the functorial properties of Z, we get that

$$\begin{aligned} id_{Z(M)} &= Z((id_M \times coev_M) \circ (ev_M \otimes id_M)) \\ &= (Z(ev_M) \otimes id_M) \circ (id_{Z(M)} \otimes Z(coev_M)), \end{aligned}$$

where we identify $Z(M)$ with $Z(M) \otimes k$ and $k \otimes Z(M)$ by the canonical isomorphisms to simplify notation.

For an element $v \in Z(M)$ this equation gives

$$v = id_{Z(M)}(v) = (Z(ev_M) \otimes id_{Z(M)})(v \otimes Z(coev_M)(1)$$



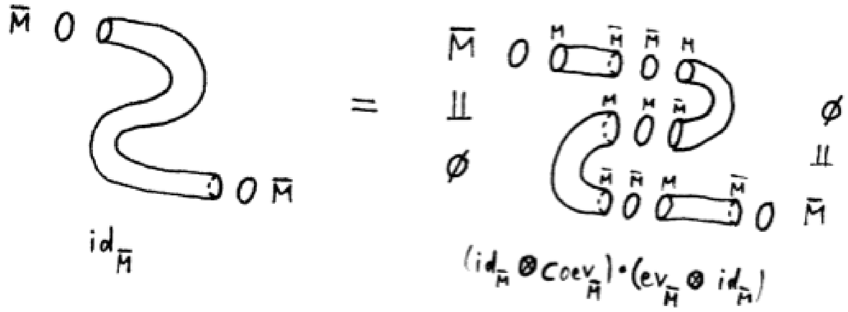
$$= (Z(ev_M) \otimes id_{Z(M)})(v \otimes \sum_{i=1}^{n} w_i \otimes v_i)$$

$$= (Z(ev_M) \otimes id_{Z(M)})(\sum_{i=1}^{n} v \otimes w_i \otimes v_i)$$

$$= \sum_{i=1}^{n} Z(ev_M)(v \otimes w_i)v_i$$

$$= \sum_{i=1}^{n} \alpha(w_i)(v)v_i. \quad (1)$$

Now we do the same for the cobordism $id_{\bar{M}}$ and its decomposition into $id_{\bar{M}} \otimes coev_{\bar{M}}$ and $ev_{\bar{M}} \otimes id_{\bar{M}}$ as is shown in the picture below.



This gives us the equation

$$id_{\bar{M}} = Z((id_{\bar{M}} \otimes coev_{\bar{M}}) \circ (ev_{\bar{M}} \otimes id_{\bar{M}}))$$

$$= (Z(ev_{\bar{M}}) \otimes id_{Z(\bar{M})}) \circ (id_{Z(\bar{M})} \otimes Z(coev_{\bar{M}})).$$

Using $coev_{\bar{M}} = coev_M \circ \text{swap}_{\bar{M},M}$ and $ev_{\bar{M}} = \text{swap}_{\bar{M},M} \circ ev_M$ and the symmetry of Z this leads to

$$\begin{aligned} w = id_{Z(\bar{M})}(w) &= (Z(ev_{\bar{M}}) \otimes id_{Z(\bar{M})})(w \otimes Z(coev_{\bar{M}})(1)) \\ &= (Z(ev_{\bar{M}}) \otimes id_{Z(\bar{M})})(w \otimes (\sum_{i=1}^{n} v_i \otimes w_i)) \\ &= (Z(ev_{\bar{M}}) \otimes id_{Z(\bar{M})})(\sum_{i=1}^{n} w \otimes v_i \otimes w_i) \\ &= \sum_{i=1}^{n} Z(ev_{\bar{M}})(w \otimes v_i) w_i \\ &= \sum_{i=1}^{n} Z(ev_M)(v_i \otimes w) w_i \\ &= \sum_{i=1}^{n} \alpha(w)(v_i) w_i. \end{aligned} \tag{2}$$

With these two equations we can show that β defined above is in fact the inverse of α:

Let $w \in Z(\bar{M})$. Then $\alpha(w)(v) = Z(ev_M)(v \otimes w)$ and $\beta(\alpha(w)) = \sum_{i=1}^{n} \alpha(w)(v_i) w_i = w$ by (2).

Let $\omega \in Z(M)^*$. Then $\beta(\omega) = \sum_{i=1}^{n} \omega(v_i) w_i$. Using the linearity of $Z(ev_M)$ and ω, we get

$$\begin{aligned} \alpha(\beta(\omega))(v) &= Z(ev_M)(v \otimes \sum_{i=1}^{n} \omega(v_i) w_i) \\ &= \sum_{i=1}^{n} \omega(v_i) Z(ev_M)(v \otimes w_i) \\ &= \omega\left(\sum_{i=1}^{n} Z(ev_M)(v \otimes w_i) v_i\right) \\ &= \omega(\alpha(w_i)(v) v_i) \\ &= \omega(v), \end{aligned}$$

hence $\alpha(\beta(\omega)) = \omega$, which implies that $\beta = \alpha^{-1}$.

We are now left to show that $Z(M)$ is finite dimensional for every object M of $nCob$. For this, let us consider the map $\beta : Z(M)^* \to Z(\bar{M})$. If we look at the construction of β we see that the image of β is generated by the vectors $w_1, \ldots, w_n$ in $Z(\bar{M})$. Since β is surjective, these vectors generate

$Z(\bar{M})$, which therefore is finite dimensional. Since this is true for any object M in $nCob$, and $\bar{\bar{M}} = M$, we get that $Z(M)$ is finite-dimensional for every object M in $nCob$, which completes the proof of Proposition 4. $\square$

We will now use this proposition and the description of $1Cob$, to completely describe all 1TFTs.

Let Z be a 1TFT. Denote $V := Z(P)$ and $W := Z(Q)$. Proposition 4 gives us a canonical isomorphism between W and V^*. Then for any object $M = \coprod_{i=1}^k P \coprod \coprod_{j=1}^l Q$, we have

$$Z(\coprod_{i=1}^k P \coprod \coprod_{j=1}^l Q) \simeq V^{\otimes k} \otimes W^{\otimes l} \simeq V^{\otimes k} \otimes (V^*)^{\otimes l},$$

where in the last step we applied the isomorphism α to every W in the tensor product. So we know what Z does with the objects.

Since every cobordism is a finite union of connected cobordisms and $Z([B, \imath] \coprod [B', \imath'])$ corresponds to $Z([B, \Phi]) \otimes Z([B', \Phi'])$ under the canonical isomorphisms coming along with Z, we only need to know where Z sends the connected cobordism classes. Using our knowledge about the cobordism $\mathrm{swap}_{P,Q}$ and $Z(\mathrm{swap}_{P,Q})$ we find that we only need to understand the following cobordisms:

1) 

$\mapsto id_V : V \to V$

2) 

$\mapsto id_W : W \to W$

3) 

$\mapsto Z(ev_P) : V \otimes W \to k$

4) 

$\mapsto Z(coev_P) : k \to W \otimes V$

5) 

$\mapsto Z(coev_P \circ ev_Q) : k \to k$

The first two maps are just the identities on V respectively W.

In order to understand the third map we use the map α to see that $Z(ev_P)$ corresponds to a map $V \otimes V^* \to k$ which sends $v \otimes \omega$ to $Z(ev_P)(v \otimes \beta(\omega)) = \alpha(\beta(\omega))(v) = \omega(v)$. Therefore $Z(ev_P)$ corresponds to the evaluation map on $V \otimes V^*$ under the isomorphism $id_V \otimes \alpha$.

The fourth map is a k-linear map which is defined on the field k. Therefore it is completely described by the image of 1. Using the notation of Proposition 4, we have that $Z(coev_P) = \sum_{i=1}^{n} w_i \otimes v_i$. We now do the following identifications:

$$W \otimes V \simeq V^* \otimes V \text{ under the isomorphism } \alpha \otimes id_V.$$

$$V^* \otimes V \simeq \mathrm{End}(V) \text{ by the linear extension of } \omega \otimes v \mapsto \omega(\cdot)v.$$

If we apply these isomorphisms to $Z(coev_P)(1)$, we get

$$Z(coev_P)(1) = \sum_{i=1}^{n} w_i \otimes v_i \mapsto \sum_{i=1}^{n} \alpha(w_i) \otimes v_i \mapsto \sum_{i=1}^{n} \alpha(w_i)(\cdot)v_i.$$

Using equation (2), we see that the resulting endomorphism is the identity map on V, hence if we consider $Z(coev_P)$ as a map from k to $\mathrm{End}(V)$, we get that it is the map $Z(coev_P)(x) = x \cdot id_V$.

To understand the fifth map we look at S^1 as the composition $coev_P \circ \mathrm{swap}_{Q,P} \circ ev_P$. Thus we see that $Z(S^1) = Z(ev_P) \circ swap_{W,V} \circ Z(coev_P)$. This map is determined by $Z(S^1)(1) = Z(ev_P)(\sum_{i=1}^{n} v_i \otimes w_i)$. We know that the composition of $id_V \otimes \alpha$ and the canonical identification of $V \otimes V^*$ with $\mathrm{End}(V)$ sends $\sum_{i=1} v_i \otimes w_i$ to id_V. Denote this last identification by $\psi : V \otimes V^* \to \mathrm{End}(V)$. Then $\psi^{-1}(id_V) = \sum_{i=1}^{\dim(V)} e_i \otimes e_i^*$, where $e_1, \ldots, e_{\dim(V)}$ is a basis of V and $e_1^*, \ldots, e_{\dim(V)}^*$ its dual basis. Therefore

$$Z(ev_P)(\sum_{i=1}^{n} v_i \otimes w_i) = Z(ev_P)(\sum_{i=1}^{\dim(V)} e_i \otimes e_i*) = \sum_{i=1}^{\dim(V)} e_i^*(e_i) = \dim(V).$$

This implies that $Z(S^1)(x) = x \dim(V)$ and hence, the linear map $Z(S^1) : k \to k$ corresponds to the number $\dim(V) \in k$.

Using our knowledge about these five cobordisms and $\mathrm{swap}_{P,Q}$ we can describe $Z([B, \imath])$ for any cobordism class $[B, \imath]$ as a finite tensor product of the maps discussed above composed with some applications of $\mathrm{swap}_{V,V}$, swap_{V^*,V^*}, swap_{V,V^*} and $\mathrm{swap}_{V^*,V}$.

Therefore we see, that Z is completely determined by the vector space $Z(P)$ which is determined up to vector space isomorphism by its dimension $\dim(Z(P))$. Thus, every 1TFT is described by its image of S^1.

4 References

1. On the Classification of Topological Field Theories, Jacob Lurie, Current developments in mathematics, 2008, 129-280, Int. Press, Sommerville, MA, 2009.