Duality in monoidal categories

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Left and right duals in a monoidal category

We recall the concept of the dual space of a vector space:

Definition 1 (Dual vector space)**.** *Let V be a k-vector space. Then its dual space (denoted by $V^\vee$) is the k-vector space of linear functions $V \to k$.*

The essential feature of finite-dimensional vector spaces is a well-behaved duality theory. In fact, if V is a finite-dimensional vector space on a field k, we can define two canonical (linear) maps, the evaluation $\mathrm{ev}_V : V \otimes V^\vee \to k$, $(v, \lambda) \mapsto \lambda(v)$ and the coevaluation map $\mathrm{coev}_V : k \to V^\vee \otimes V$, which is completely determined by $\mathrm{coev}_V(1) = \epsilon^i \otimes e_i \cong \mathrm{id}_V$ (here $\{\epsilon^i\}_i$, $\{e_i\}_i$ are bases of $V^\vee$ and V respectively; coev_V is independent from the choice of such bases, as you can easily check).
These two construction are "inverse" one to each other, in the sense that they satisfy the following compatibility relations $(\mathrm{ev}_V \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}_V) = \mathrm{id}_V$ and $(\mathrm{id}_{V^\vee} \otimes \mathrm{ev}_V) \circ (\mathrm{coev}_V \otimes \mathrm{id}_{V^\vee}) = \mathrm{id}_{V^\vee}$. We check the first equation:
Fix bases $\{\epsilon^i\}_i$, $\{e_i\}_i$ for $V^\vee$ and V, let $v = v^i e_i \in V$. By direct calculation we have:

$$(\mathrm{ev}_V \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \mathrm{coev}_V)(v) = (\mathrm{ev}_V \otimes \mathrm{id}_V)(v^i e_i \otimes \epsilon^j \otimes e_j) = v^i \delta_i^j e_j = v$$

So it is indeed equal to id_V. The second equation can be checked similarly.

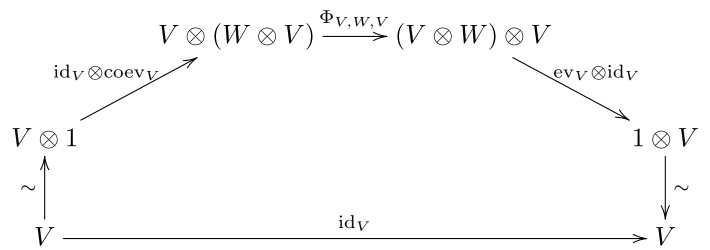
We want to generalize this concept of well behaved duality ($V^{\vee\vee} = V$) to monoidal categories. We do so by rephrasing the concept of dual objects using analogues of the evaluation and coevaluation maps.

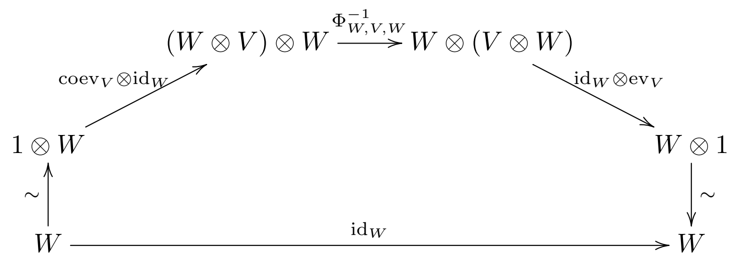
Definition 2 (Left and right duals)**.** *Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$ with $1 \in \mathcal{C}$ as unit. Let V be an object of $\mathcal{C}$. We say that an object W of $\mathcal{C}$ is a right dual of V it there exist maps*

$$\mathrm{ev}_V : V \otimes W \to 1$$

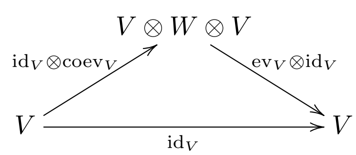
$$\mathrm{coev}_V : 1 \to W \otimes V$$

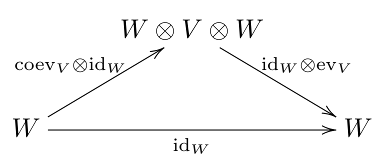
such that the following diagrams commute:





Where $\Phi_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C$ is the isomorphism given by associativity of the tensor product.
Since the isomorphisms Φ and $V \xrightarrow{\sim} V \times 1$ are unique, we will from now on omit them from the diagrams and write:





We also say that V is a left dual of W. If W is both a left and a right dual of V, we simply say that W is a dual of V and denote it by $V^\vee$.
An object of $\mathcal{C}$ having a dual is called dualizable.

Remark 3. *Let $\mathcal{C}$ be a monoidal category and V a dualizable object of $\mathcal{C}$. Let W be a dual of V. Then W is also dualizable and V is a dual of W.*

Example 4. *All finite dimensional vector spaces are dualizable in the category* $\mathbf{Vect(k)}$ *with evaluation and coevaluation maps as defined in the beginning.*

Remark 5. *There are categories where right and left duals do not coincide. We will not present examples of this fact.*

Duals are unique, if they exist. We state this in a lemma.

Lemma 6. *Let $\mathcal{C}$ be a monoidal category and let V be an object in $\mathcal{C}$. Let W be a right (left) dual of V. Then W is the unique (up to isomorphisms) right (resp. left) dual of V.*

Proof. We show this holds for right duals. The proof for left duals is very similar and the case of duals is an obvious consequence of the validity of the lemma for right and left duals.
Let V, W and W' be objects in $\mathcal{C}$, and assume that both W and W' are right duals of V. We will construct an isomorphism between W and W'.
The evaluation map leads to a natural isomorphism

$$\alpha : \hom(-,W) \overset{\sim}{\to} \hom(V \otimes -, 1)$$

given by $\alpha(\varphi) = \mathrm{ev}_V \circ (\mathrm{id}_V \otimes \varphi)$ (it is easy to check that its inverse is given by $\alpha^{-1}(\phi) = (\mathrm{id}_W \otimes \phi) \circ (\mathrm{coev}_V \otimes \mathrm{id}_-))$. In the same way, we can obtain a natural isomorphism $\alpha' : \hom(-,W') \overset{\sim}{\to} \hom(V \otimes -, 1)$. This way, we can construct another natural isomorphism $\beta_- : \hom(-,W) \overset{\sim}{\to} \hom(-,W')$ by $\beta_- = (\alpha')^{-1} \circ \alpha$.
Consider $f = \beta_W(\mathrm{id}_W) \in \hom(W,W')$ and $g = \beta^{-1}_{W'}(\mathrm{id}_{W'}) \in \hom(W',W)$. We get the following (obviously) commuting diagram:

$$\begin{array}{ccccc}
\mathrm{id}_W & \longleftarrow & & \longmapsfrom & f \\
 & & & & \uparrow \\
\hom(W,W) & \xrightarrow{\beta_W} & \hom(W,W') & & \\
\uparrow{\scriptstyle \circ f} & & \uparrow{\scriptstyle \circ f} & & \\
\hom(W',W) & \xrightarrow[\beta_{W'}]{} & \hom(W',W') & & \\
 & & & & \Big| \\
g & \longmapsto & & \longrightarrow & \mathrm{id}_{W'}
\end{array}$$

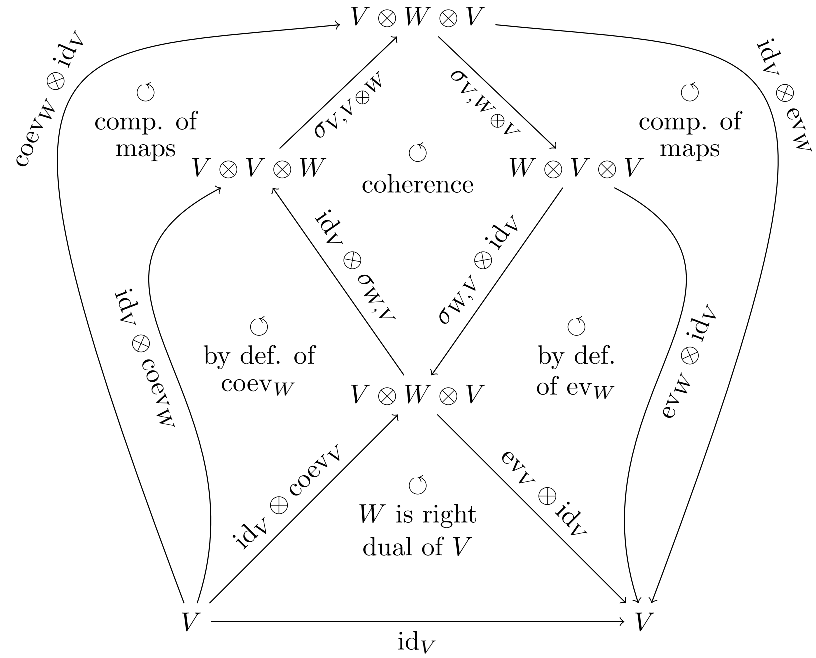
From this we see that $g \circ f = \mathrm{id}_W$. Similarly, we can get $f \circ g = \mathrm{id}_{W'}$. Thus f and g are inverses one to another and form isomorphisms between W and W'. $\square$

We also have the following useful property for symmetric monoidal categories:

Lemma 7. *Let $\mathcal{C}$ be a symmetric monoidal category, V an object in $\mathcal{C}$. If W is a left (right) dual of V, then it is also a right (left) dual of V, and thus a dual.*

Proof. Let $\sigma_{A,B} : A \otimes B \to B \otimes A$ be the commutativity constraint. Since $\mathcal{C}$ is a symmetric monoidal category, we have $\sigma_{B,A} \circ \sigma_{A,B} = \mathrm{id}_{A \otimes B}$.
Without loss of generality, we will treat only the case where W is a right dual.

We will show that $\mathrm{ev}_W := \mathrm{ev}_V \circ \sigma_{V,W}$ and $\mathrm{coev}_W := \sigma_{W,V} \circ \mathrm{coev}_V$ are the evaluation and coevaluation maps for W. We show that the axioms for a right dual hold by obtaining the second axiom for the statement "V is a right dual of W" from the first axiom for "W is a right dual of V" (and vice-versa). We do so by means of the following commuting diagram:



Here we have omitted the associativity constraints.
The other diagram is similar. □

Examples

Example 8 (Dualizability in **Sets**)**.** *We treat the case of the symmetric monoidal category* **Sets**. *Here the tensor product (which is simply the product of sets* $\times$*) has one-point sets as unit (uniqueness is given by the fact that they are all canonically isomorphic to one another, and we denote the one point set by* $*$*). We show that the only dualizable object of* **Sets** *is* $*$*, and that it is the dual of itself.*
Assume a set S *in* **Sets** *has a dual* T*. Then* $\mathrm{id}_S = (\mathrm{ev}_S \times \mathrm{id}_S) \circ (\mathrm{id}_S \times \mathrm{coev}_S)$*. The image of* coev_S *contains exactly one element, since* coev_S *takes arguments in a one-point set, and thus it is* $*$*. Thus*

$$S = \mathrm{id}_S(S) = (\mathrm{ev}_S \times \mathrm{id}_S) \circ (\mathrm{id}_S \times \mathrm{coev}_S)(S) = (\mathrm{ev}_S \times \mathrm{id}_S)(S \times * \times *) = *$$

It is easy to see that $\mathrm{ev}_* : * \times * \mapsto *$ *and* $\mathrm{coev}_* : * \mapsto * \times *$ *satisfy both axioms for a dual.*

Example 9 (Dualizability in $\mathbf{Vect(k)}$)**.** *We can show that an object of* $\mathbf{Vect(k)}$ *is dualizable if, and only if, it is finite dimensional.*
Assume $V \in \mathbf{Vect(k)}$ *has finite dimension. Since the tensor product of vector spaces is symmetric, by lemma 7 it is enough to prove that it has a right dual. This dual is given by the k-vector space of linear functions* $V \to k$ *with evaluation and coevaluation maps as defined in the beginning.*
Now assume that an object V *of* $\mathbf{Vect(k)}$ *has a dual* W*. Proceeding like we did in the last example, we notice that the image of* coev_V *in* $W \otimes V$ *is 1-dimensional, and thus the image of* id_V *is finite-dimensional.*

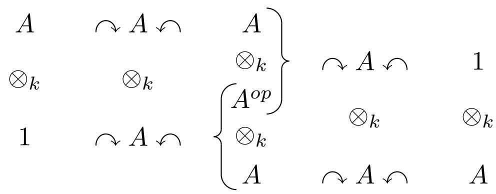
Remark 10. *Note that every k-vector space* V *has a linear dual (the vector space of linear functions from* V *to* k*), but the notion of linear dual is not equal to the concept of dual given by definition 2. In fact, in the category* $\mathbf{Vect(k)}$*, a dual is a linear dual, but the converse doesn't hold.*

Example 11 (Dualizability in $\mathbf{R-mod}$)**.** *The category* $\mathbf{R-mod}$ *is composed by R-modules as objects, R-module homomorphisms as morphisms and tensor product of R-modules, denoted by* $\otimes_R$ *as monoidal structure (with* R *as unit).*
The dualizable objects in $\mathbf{R-mod}$ *are all finitely generated projective R-modules. We will not prove this fact.*

Example 12 (Dualizability in $\mathbf{Alg(k)}$)**.** *First of all, let us recall how the symmetric monoidal category* $\mathbf{Alg(k)}$ *is composed. The objects of this category are the associative k-algebras (not necessarily commutative, while the morphisms from an object* A *to an object* B *are the isomorphism classes of* (A, B)*-bimodules, i.e. modules with* A *acting on the right and* B *acting on the left, such that* $(a \cdot m) \cdot b = a \cdot (m \cdot b)$*.*
The composition of morphisms is defined as follows:
given two morphisms $A \xrightarrow{M} B \xrightarrow{N} C$*, the composition of* N *with* M *is the tensor product over* B *of the two bimodules, i.e.* $N \otimes_B M = (N \otimes M)/\sim$*, where the equivalence is given by* $(m \cdot b) \otimes n \sim m \otimes (b \cdot n)$*.*
The monoidal structure is given simply by the tensor product over k*, which we denote by* $\otimes_k$*. The identity element is* $k = 1$*.*
We show that every object A *of* $\mathbf{Alg(k)}$ *is dualizable, and that* A^{op} *("A opposite", denotes* A *with inverse multiplication).*
We can interpret A *as a module in four important different ways:*

- *A as a (A, A)-module (both copies of A act by multiplication, the first copy on the left, the second on the right): with this interpretation, A is the identity morphism, denoted by id_A.*
- *A as a (A^{op}, A^{op})-module, which gives us $\mathrm{id}_{A^{op}}$.*
- *A as a $(A \otimes_k A^{op}, 1)$-module. We denote this morphism by ev_A.*
- *A as a $(1, A^{op} \otimes_k A)$-module. We denote this morphism by coev_A.*

We can check that ev_A *and* coev_A *are in fact the evaluation and coevaluation maps in the sense of definition 2. The following diagram gives us an idea about* *how this works:*



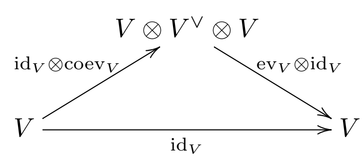
We can see from the drawing above that the composition of morphisms gives us id_A.

Application to nCob and TFTs

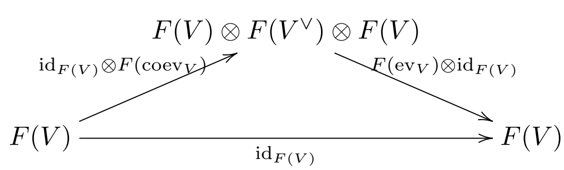
One of the useful properties of dualizability is its nice behaviour under symmetric monoidal functors.

Lemma 13. *Let $\mathcal{C}$ and $\mathcal{D}$ be two symmetric monoidal categories and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Let V be a dualizable object of $\mathcal{C}$ with dual $V^\vee$. Then $F(V)$ is dualizable and $F(V^\vee)$ is a dual of $F(V)$. Moreover, the evaluation and coevaluation maps for this dual are given by $\mathrm{ev}_{F(V)} = F(\mathrm{ev}_V)$ and $\mathrm{coev}_{F(V)} = F(\mathrm{coev}_V)$.*

Proof. First of all, it is important to note that, by lemma 7, it is sufficient to prove that the image of a right dual of V under F is a right dual of $F(V)$ to prove the assertion.
We see easily that the diagrams giving the axioms for a dual in $\mathcal{D}$ are simply the image of the diagrams of the axioms for a dual in $\mathcal{C}$ under the symmetric monoidal functor F.
We show this for the first of the two axioms. Let V be an object of $\mathcal{C}$ and $V^\vee$ a dual. Keeping in mind that, by definition, $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$ and $F(f \otimes g) = F(f) \otimes F(g)$ for all morphisms f and g in $\mathcal{C}$, we have that the commuting diagram



is sent to the (still commuting) diagram



The other axiom is treated similarly. We can then conclude that $F(V^\vee)$ is dual to $F(V)$, $\mathrm{ev}_{F(V)} = F(\mathrm{ev}_V)$ and $\mathrm{coev}_{F(V)} = F(\mathrm{coev}_V)$. □

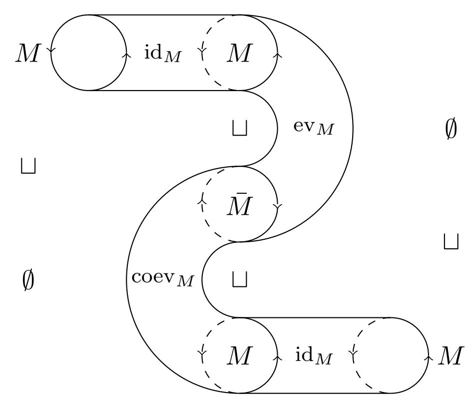
The dualizability of objects in the category **nCob** is very important in the study of TFTs. It is also very well behaved. We state it in a lemma:

Lemma 14 (Dualizability in **nCob**). *Every object in* **nCob** *is dualizable.*

Proof. Recall that **nCob** is a symmetric monoidal category with the disjoint union $\sqcup$ as tensor product and the empty set $\emptyset$ as identity. By lemma 7, we only need to show that every object in **nCob** has a right dual.
Let M be an object in **nCob**, that is an oriented $(n-1)$-manifold. We denote by $\bar{M}$ the same manifold with reverse orientation. We will show that $\bar{M}$ is a right dual for M. Let the evaluation and coevaluation maps be defined as follows:

- $M \sqcup \bar{M} \stackrel{\mathrm{ev}_M}{\to} \emptyset$ is given by the cobordism $M \times [0,1]$ interpreted as cobordism from $M \sqcup \bar{M}$ to $\emptyset$.
- $\emptyset \stackrel{\mathrm{coev}_M}{\to} \bar{M} \sqcup M$ is given by the cobordism $M \times [0,1]$ interpreted as cobordism from $\emptyset$ to $\bar{M} \sqcup M$.
- We also remember that $M \times [0,1]$ interpreted as cobordism from M to $\bar{M}$ gives us the identity morphism id_M and that $M \times [0,1]$ interpreted as cobordism from $\bar{M}$ to M gives us the identity morphism $\mathrm{id}_{\bar{M}}$.

An intuition on how to prove that ev_M and coev_M effectively satisfy the axioms for a dual is given by the following drawing:



In a somewhat more formal way, we can identify $\mathrm{id}_M \otimes \mathrm{coev}_M$ with the cobordism $(M \times [0,1]) \sqcup (M \times [2,3])$ and $\mathrm{ev}_M \otimes \mathrm{id}_M$ with the cobordism $(M \times [1,2]) \sqcup (M \times [3,4])$. The composition of morphisms then identifies $M \times \{i\}$ with $M \times \{i\}$, for $i = 1,2,3$. This gives us an n-manifold equivalence class diffeomorphic with $M \times [0,4]$, and thus to $M \times [0,1] = \mathrm{id}_M$. □

A direct consequence of those lemmas is: if Z : **nCob** $\rightarrow$ **Vect**(**k**) is an nTFT, then $Z(M)$ has finite dimension $\forall$ $(n-1)$-manifold M object of **nCob**. This is given by the fact that M has $\bar{M}$ as a dual (by lemma 14) and that thus $Z(\bar{M})$ is dual to $Z(M)$ (by lemma 13) and this is possible only if $Z(M)$ has finite dimension (as we have seen in example 9).

References

1. J. Lurie, On the classification of topological field theories, Current developements on mathematics, 2008, 129-280, Int. Press, Sommerville, MA, 2009.

2. http://unapologetic.wordpress.com/2008/11/13/the-coevaluation-on-vector-spaces/, 23 September 2012.