Generators and relations in 1Cob

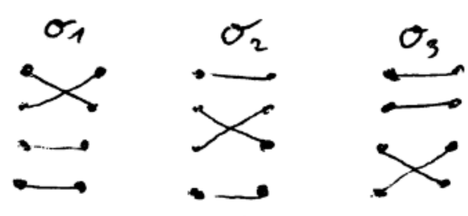
Julian Zimmert

Generators and relations of a group

Definition 1 (Generators of a group). *A generating set of a group is a subset such that every element of the group can be expressed as a finite combination (under the group operation) of finitely many elements of the subset and their inverses.*

Definition 2 (Relations of a group). *A relation is the equivalence of writing an element in terms of the generators*

Example 3 (Symmetric). *The Symmetric group $\mathcal{S}_4$ has the generators σ_1, σ_2, σ_3.*



With the relations

$$\sigma_i^2 = id$$

$$\sigma_i\sigma_j = \sigma_j\sigma_i \ , ifj \neq i \pm 1$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma i\sigma i + 1$$

Generators and relations of a monoidal category

Definition 4 (Skeleton). *A skeleton of a category $\mathcal{C}$ is a category D satisfying the following:*

- *Every object in D is an object in $\mathcal{C}$.*
- *For every pair of objects X and Y in D, the arrows in D are precisely the arrows in C, thus $D(X,Y) = \mathcal{C}\ (X,Y)$.*

- *For every object X in D, its identity arrow with respect to D coincides with its identity with respect to $\mathcal{C}$.*
- *The composition law of arrows in D equals the composition law of arrows in $\mathcal{C}$, when restricted to arrows in D.*
- *Every object in $\mathcal{C}$ is isomorphic to some object in D.*
- *There exists no isomorphism between any pair of distinct objects in D.*

Definition 5 (Generators and relations of a (symmetric) monoidal category)**.**
Let D be a skeleton of our category.
A generating set $G(D)$ is a collection of arrows in D so that every other arrow in D can be obtained by composing arrows in $G(D)$.
(by composing we mean the normal composition and the "tensoring" our in our case "paralleling")
This set is not necessarily unique, but we assume it is minimal.
Any arrow in $G(D)$ is called a generator . A relation is the equality of two distinct ways of writing a given arrow in terms of these generators. A minimal set $R(D)$ of relations is called complete if every other relation can be obtained by combining relations in $R(D)$.

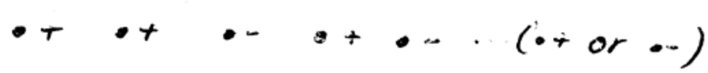
Finding a skeleton of 1Cob

- First: finding an "almost skeleton" 1Cob' and giving the generators
- Second: reducing it to 1*cob* a proper skeleton and making sure it's a symmetric monoidal category

1Cob'

objects

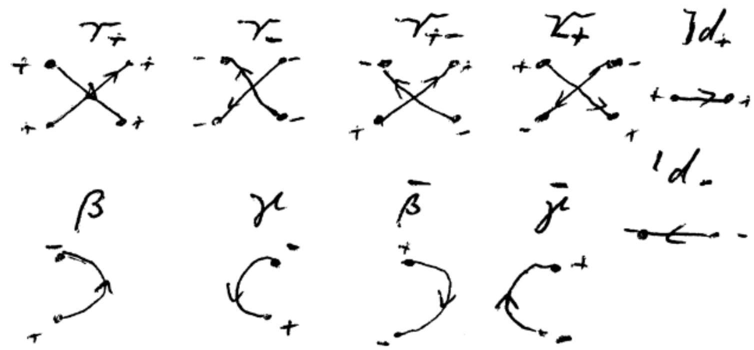
If we choose an arbitrary point p without orientation, then p_+ and p_- are oriented points. We can use these two points as basic objects in 1Cob'. Our Objects in 1Cob' are disjoint unions of the basic objects.



$$p_+ \sqcup p_+ \sqcup p_- \sqcup p_+ \sqcup ...$$

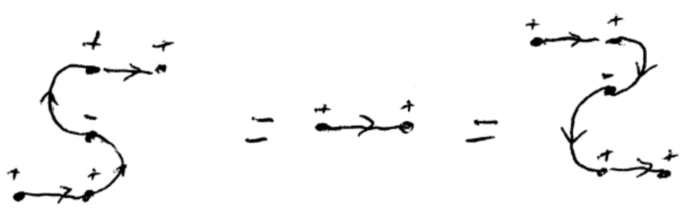
generators

The generators of 1Cob' are $\tau_+, \tau_-\tau_{+-}, \tau_{-+}, id_+, id_-, \beta, \gamma, \overline{\beta}, \overline{\gamma}$



snake relation

$$(\overline{\gamma} \sqcup id_+)(id_+ \sqcup \beta) = id_+ = (id_+ \sqcup \gamma)(\overline{\beta} \sqcup id_+)$$



monoidal structure

We can just take the disjoint union and composition from 1Cob, because it is closed in 1Cob.

1Cob' is not a skeleton because τ_{+-} and $\tau - +$ are isomorphism between distinct objects.

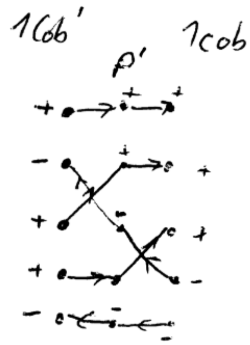
1Cob

objects

We take only the ordered objects of 1cob, in other words a disjoint union of an only positive with an only negative object.

$$(p_+)^k \sqcup (p_-)^n$$

There is a trivial projektion $P' : 1Cob' \to 1cob$



monoidal structure

1cob is not closed in terms of the disjoint union of 1Cob, so we need to make some adjustements:

$$X, Y \in 1cob \subset 1Cob'$$

$$X \sqcup' Y = P'(X \sqcup Y)$$

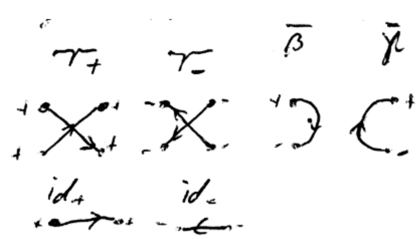
Define $l_{X \sqcup Y} := id_{X_+} \sqcup \tau_{x_-, Y_+} \sqcup id_{Y_-}$
The disjoint union of arrows is now:

$$f : A \to B, g : C \to D, f, g \in 1cob$$

$$f \sqcup' g := P_1(f \sqcup g) = (f \sqcup g)' = l_{B \sqcup D}(f \sqcup g){l_{A \sqcup C}}^{-1}$$

generators

The only generators from 1Cob' that are now left are:
$\tau_+, \tau_-, \overline{\beta}, \overline{\gamma}$



snake relation

Using the projector on the snake composition in Cob1' leads to

$$id_- = (\overline{\gamma} \sqcup id_-)(id_+ \sqcup \tau_-)(\overline{\beta} \sqcup id_-)$$

$$id_+ = (id_+ \sqcup \overline{\beta})(\tau_+ \sqcup id_-)(id_+ \sqcup \overline{\gamma})$$