Two-dimensional topological field theories and Frobenius algebras

Seminar on Higher Homotopical Algebra

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1 Frobenius Algebras

Frobenius algebras were first defined as algebras over some field together with a nondegenerate bilinear form or, equivalently, a trace. As it turns out it is possible to generalize this concept to categories.

Categorical approach

A Frobenius object in a category is an object together with a collection of four morphisms satisfying certain axioms. The axioms assume a (symmetric) monoidal structure on the category. We refer the reader who is not familiar with these categories to (Mac Lane 1971, Chapter VII).

(Co-)monoid objects. Let $(\mathbf{C}, \square, \mathbf{1})$ be a monoidal category. A *monoid object* is an object A together with a morphisms $\beta : A \square A \to A$ called multiplication and a morphism $g : \mathbf{1} \to A$ called unit, such that the following diagrams commute.

$$\begin{array}{ccc} (A \,\square\, A) \,\square\, A & \xleftrightarrow{\ \sim\ } & A \,\square\, (A \,\square\, A) \\ \Big\downarrow{\scriptstyle \beta \,\square\, \mathrm{id}} & & \Big\downarrow{\scriptstyle \mathrm{id} \,\square\, \beta} \\ A \,\square\, A & & A \,\square\, A \\ & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & A & \end{array} \tag{A}$$

$$\begin{array}{ccccc} A \,\square\, \mathbf{1} & \xrightarrow{\ \mathrm{id} \,\square\, g\ } & A \,\square\, A & \xleftarrow{\ g \,\square\, \mathrm{id}\ } & \mathbf{1} \,\square\, A \\ & {\scriptstyle \sim}\searrow & \Big\downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle \sim} & \\ & & A & & \end{array} \tag{U}$$

We say that β is an associative multiplication with unit g.

Assume the category is equipped with a symmetry structure τ. The multiplication is said to be *commutative* if the diagram

$$\begin{array}{ccc} A \,\square\, A & \xleftrightarrow{\ \tau_{AA}\ } & A \,\square\, A \\ & {\scriptstyle \beta}\searrow \quad \swarrow{\scriptstyle \beta} & \\ & A & \end{array} \tag{C}$$

commutes.

We can dualize the notion of a monoid object. Let A be an object with morphisms $\alpha : A \to A \,\square\, A$ and $f : A \to \mathbf{1}$. We say that (A, α, f) is a *co-monoid* if it is a monoid in $\mathbf{C}^{\mathrm{op}}$. This means diagrams (A) and (U) with their arrows reversed and α and f replacing β and g commute. We call (A, α, f, β, g) a *bi-monoid* if (A, α, f) is a co-monoid and (A, β, g) a monoid.

We immediately observe, that in the category of $\mathbf{k}$-vector spaces, these are just the definitions of algebras, co-algebras and bi-algebras.

Frobenius relation. A bi-monoid is called a *Frobenius object* if it additionally satisfies the Frobenius relation, that is the following diagrams commute

$$\begin{array}{ccc} A \,\square\, A & \xrightarrow{\alpha \,\square\, \mathrm{id}} & A \,\square\, A \,\square\, A \\ {\scriptstyle \beta}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id} \,\square\, \beta} \\ A & \xrightarrow{\alpha} & A \,\square\, A \end{array} \qquad \begin{array}{ccc} A \,\square\, A & \xrightarrow{\mathrm{id} \,\square\, \alpha} & A \,\square\, A \,\square\, A \\ {\scriptstyle \beta}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta \,\square\, \mathrm{id}} \\ A & \xrightarrow{\alpha} & A \,\square\, A. \end{array} \tag{F}$$

It is clear, that symmetric monoidal functor preserves Frobenius objects.

Frobenius objects for vector spaces

Recall that in the symmetric monoidal category **Vect** of finite-dimensional vector spaces over some field **k**, monoid objects are algebras. We show that there is a more natural characterisation of Frobenius objects in **Vect** involving a linear functional, called trace or Frobenius form.

Theorem 1. Let (A, β, g) be a finite dimensional algebra and $f : A \to \mathbf{k}$ a homomorphism. The following are equivalent.

(i) There exists a co-multiplication α such that (A, α, f, β, g) is a Frobenius object.

(ii) The map $f\beta$ is non-degenerate.

Moreover, if a co-multiplication as in (i) exists, it is unique.

The theorem and its proof are a generalisation of (Abrams 1996, Theorem 1).

Proof. We first note, that β induces a right A-module structure on A and $A \otimes A$ given by

$$\begin{aligned} \beta : A \otimes A &\longrightarrow A \\ a \otimes b &\longmapsto ab \end{aligned}$$

and

$$\begin{aligned} \mathrm{id} \otimes \beta : A \otimes (A \otimes A) &\longrightarrow A \otimes A \\ a \otimes b \otimes c &\longmapsto a \otimes bc \end{aligned}$$

Similarily, a left A-module structure is defined by β and $\beta \otimes \mathrm{id}$. The proof is based on the observation that α satifsfies the Frobenius relation if and only if it is a right and left A-module homorphism.

Assume first that $f\beta$ is non-degenerate. This yields an isomorphism $\lambda : A \to A^*$ given by

$$\lambda(a)(b) = f\beta(a \otimes b) = f(ab).$$

Indeed, λ is linear and, since $f\beta$ is non-degenerate, injective. By assumption A is finite dimensional, so λ is also surjective. Tensoring λ with itself we also obtain an isomorphism

$$\lambda \otimes \lambda : A \otimes A \longrightarrow A^* \otimes A^*.$$

We claim that both isomorphisms are also right A-module homomorphisms. Here the module structure on A^* and $A^* \otimes A^*$ is given as follows. For $l \in A^*$ and $a, b \in A$ define

$$(l \cdot a)(b) = l(ab)$$

and for $l \in A^* \otimes A^*$ and $a, b, c \in A$ set

$$(l \cdot a)(b \otimes c) = l(b \otimes ac).$$

Then, for all elements a, b and c in A we see that

$$\begin{aligned}(\lambda(a) \cdot b)(c) &= \lambda(a)(bc) \\ &= f(abc) \\ &= \lambda(ab)(c).\end{aligned}$$

Hence λ is a right A-module homomorphism. A similar argument shows that $\lambda \otimes \lambda$ also is a module homomorphism.

We now dualize the algebra structure on A to obtain a coalgebra structure $(A^*, \bar{\beta}^*, g^*)$, where

$$\bar{\beta}(ab) = \beta(ba)$$

is multiplication with the operands interchanged. It follows that the comultiplication $\bar{\beta}$ is a right A-module homomorphism. Indeed, for $l \in A^*$

$$\begin{aligned}((\bar{\beta}^* l) \cdot a)(v \otimes w) &= \bar{\beta}^* l(v \otimes aw) \\ &= l\bar{\beta}(v \otimes aw) \\ &= l(awv) \\ &= (l \cdot a)(wv) \\ &= (\bar{\beta}^*(l \cdot a))(v \otimes w).\end{aligned}$$

We use λ to pull the co-algebra structure on A^* back to A by defining

$$\alpha = \left(\lambda^{-1} \otimes \lambda^{-1}\right) \circ \bar{\beta}^* \circ \lambda \; : \; A \longrightarrow A \otimes A$$

and

$$\tilde{f} = g^* \circ \lambda \,:\, A \longrightarrow \mathbf{k}.$$

The fact that $\bar{\beta}^*$ and λ are right A-module homomorphisms implies that α also is a right A-module homomorphism.

A similar argument shows that $\bar{\lambda}$ given by

$$\bar{\lambda}(a)(b) = f\bar{\beta}(a \otimes b)$$

and β^* are left A-module homomorphisms. We also note that

$$\alpha = \left(\bar{\lambda}^{-1} \otimes \bar{\lambda}^{-1}\right) \circ \beta^* \circ \bar{\lambda}$$

and conclude that α is a left A-module homomorphism. Hence the Frobenius relation (F) holds.

It remains to check that $\tilde{f} = f$. This follows from $g^*(l) = l(1_A)$ for $l \in A^*$ and

$$\tilde{f}(a) = \lambda(a)(1_A) = f(a).$$

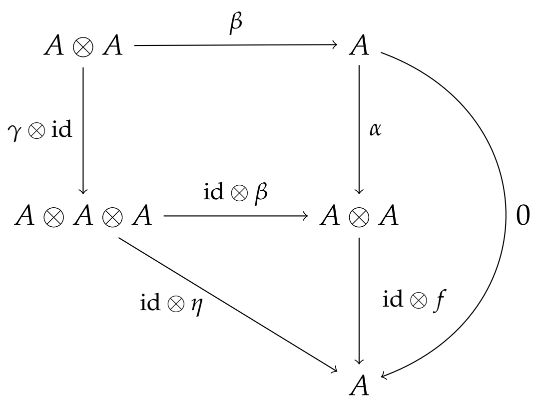
To show that α is uniquely defined by f and β, assume that α and $\tilde{\alpha}$ are commultiplications both satisfying the Frobenius relation and f is a co-unit for both, that is

$$\mathrm{id} \otimes f \circ \alpha = \mathrm{id} = \mathrm{id} \otimes f \circ \tilde{\alpha}.$$

Then $\gamma = \tilde{\alpha} - \alpha$ also satisfies the Frobensius relation and moreover

$$\mathrm{id} \otimes f \circ \alpha = 0.$$

Thus the following diagram commutes

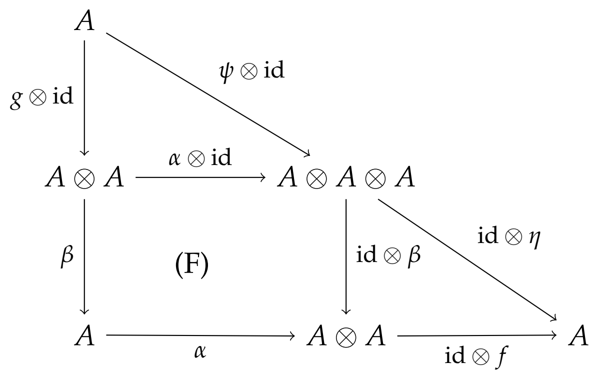


Let $\gamma(1_A) = \sum_i e_i \otimes u_i$, where the $\{e_i\}$ is a basis of A. If we evaluate the diagram from the top left corner we get

$$1_A \otimes a \mapsto \sum_i e_i \, (f\beta)(u_i \otimes a) = 0$$

for any $a \in A$. Since $f\beta$ is non-degenerate we see that $u_i = 0$ for every i. Hence $\gamma(b) = \gamma(1_A) \cdot b = 0$ for every $b \in A$ and thus $\alpha = \tilde{\alpha}$. This completes the first part of the proof.

To show the converse direction, assume a bi-aglebra structure is given on A and that the Frobenius relation holds. We define $\psi = \alpha \circ g$ and $\eta = f \circ \beta$. Then the following diagram commutes.



Using the (co-)algebra axioms the composition of the left and bottom arrows yields the identity and consequently

$$\mathrm{id}_A = \mathrm{id} \otimes \eta \circ \psi \otimes \mathrm{id}.$$

If we set $\psi(1_K) = \sum_i e_i \otimes u_i$, where the $\{e_i\}$ is a basis of A we see that for every $a \in A$

$$a = \sum_i e_i \eta(u_i \otimes a)$$

For $a = e_k$ we obtain $\eta(u_k \otimes e_k) = 1$ and conclude that η is non-degenerate. $\square$

Note that in the second half of the proof we did not use both Frobenius relations (F) but only the left one. As a corollary of the proof we thus have the following.

Corollary 1. Let A be a bi-algebra. If either one of the diagrams in (F) commutes, so does the other.

The group algebra. Let $G = \{e = g_1, g_2, \ldots, g_n\}$ be a finite group and $\mathbf{k}G$ the vector space with basis G. Multiplictaion on $\mathbf{k}G$ is defined by linearily extending the group multiplication $G \times G \to G$. The linear form f taking the values $f(g_k) = \delta_{1k}$ on the basis is a Frobenius form. Indeed, for any basis vector g_k there is $v = g_k^{-1}$ such that $f(vg_k) = f(g_1) = 1$.

2 Two-dimensional cobordisms

In this section we will develop tools to describe the category of two-dimensional oriented cobordisms in terms of a few simple building blocks, that tell us everything we need to know about the structure of this category. This will also allow us to describe any symmetric monoidal functor from this category.

The first part of the section is devoted to the definition of the symmetric monoidal category of cobordisms for an arbitrary dimension n. We then examine the special case $n = 2$, where we introduce a skeleton and discuss the implications of this reduction to the symmetric monoidal structure. We also develop a visual representation for cobordism classes. In the last part we then provide a complete representation of the morphisms in terms of generators and relations.

The oriented cobordism category

Cobordisms. A *cobordism* between two oriented $(n-1)$-dimensional manifolds X and Y is an oriented n-dimensional manifold M together with an orientation reversing embedding $i : X \to \partial M$ and an orientation preserving embedding $j : Y \to \partial M$ such that ∂M is the disjoint union of the images of i and j. We say that two cobordisms (M, i, j) and (M', i', j') are *equivalent*, if there exists an orientation preserving diffeomorphism $g : M \to M'$ such that $gi = i'$ and $gj = j'$. One can check that this is indeed an equivalence relation. The equivalence class of a cobordism will be denoted by $[M, i, j]$.

Composition. Let (M, i, j) and (M', i', j') be two cobordisms from X to Y and Y to Z respectively. We obtain the topological space $W = M \cup_h M'$ from the disjoint union $M \sqcup M'$ by identifying the subsets $j(Y)$ and $i'(Y)$ via $h = i'j^{-1}$. One can show (c.f. Milnor 1965, Theorem 1.4) that there exists an oriented manifold structure on W for which the embeddings

$\phi : M \to W$ and $\phi' : M' \to W$ are orientation preserving smooth imersions. We thus obtain a cobordism (W, ι, κ) from X to Z, where $\iota = \phi i$ and $\kappa = \phi' j'$.

We remark that the construction of this manifold W involves the choice of smooth structure, but any two such choices yield diffeomorphic structures. This means that any two cobordisms obtained this way are equivalent. Moreover the construction does not depend on the choice of equivalence classes in the first place. This means that composition is well defined on equivalence classes.

The cobordism category. The previous observations about cobordisms motivate the following definition of a category **nCob** for $n \in \mathbf{N}$. The objects are oriented closed $(n-1)$-dimensional manifolds and morphisms are equivalence classes of oriented cobordisms. The composition of two morphisms is defined as above. Of course, one has to check that composition is associative and that there exists an identity. We construct the identity in the following paragraph.

The diffeomorphism functor. Eventually, our goal is to classify the objects and morphisms in the cobordism category. As we are working with oriented manifolds, for which there are a number of classification theorems, we are looking for a way to translate them to our category. Indeed, there is a functorial relation.

Let $\phi : X \to Y$ be an orientation preserving diffemorphism. We set $M = X \times [0, 1]$ and define maps

$$\begin{aligned} i : X &\longrightarrow X \times \{0\} \\ x &\longmapsto (x, 0) \end{aligned}$$

and

$$\begin{aligned} j : Y &\longrightarrow Y \times \{1\} \\ y &\longmapsto (\phi(y), 1) \end{aligned}$$

Then $c_\phi = [M, i, j]$ is a morphism from X to Y. We have the following proposition.

Proposition 1. Let $\phi : X \to Y$ and $\psi : Z \to W$ be orientation preserving diffeomorphisms and c be the class of a cobordism (M, i, j) from Y to Z. Then

$$c_\phi c c_\psi = [M, i\phi, j\psi]$$

In particular, the map $\phi \mapsto c_\phi$ is a contravariant functor from the category of orientation preserving diffeomorphisms of oriented closed $(n-1)$-manifolds to **nCob**. Moreover, two isotopic diffeomorphisms induce the same cobordism class.

For a proof see (Milnor 1965, Theorems 1.6 and 1.9). A first consequence is the existence of an identity, induced by the identity diffeomorphism. Moreover, since every diffeomorphism is invertible, so are the cobordisms of the form c_ϕ. We also have the following important corollary.

Corollary 2. Let $c = [M, i, j]$ and $c' = [M, i', j']$ be two cobordism classes satisfying $\operatorname{im} i = \operatorname{im} i'$ and $\operatorname{im} j = \operatorname{im} j'$. Then there exist diffeomorphisms $\phi = i^{-1}i'$ and $\psi = j^{-1}j'$ such that

$$c_\phi c c_\psi = c'.$$

Symmetric Monoidal structure. Let X and Y be two compact oriented manifolds. Taking the disjoint union we obtain a new compact and oriented manifold $X \sqcup Y$ of the same dimension. We stress that it is important to distinguish between $X \sqcup Y$ and $Y \sqcup X$ as objects, altough they are naturally diffeomorphic. In a similar manner we define the disjoint union of two cobordisms

$$\begin{aligned} c\ &= [M, i, j] \quad : X \longrightarrow Y, \\ c' &= [M', i', j'] : X' \longrightarrow Y' \end{aligned}$$

Set $W = M \sqcup M'$. By composing i with the inclusion $\partial M \to \partial W$ we obtain an embedding $X \to \partial W$ that we also denote by i and similarly for i'. This induces a map

$$\iota = i \sqcup i' \ : \ X \sqcup X' \longrightarrow W.$$

Repeating the construction for j and j' we obtain a map κ. It is readily checked that (W, ι, κ) is indeed a cobordism going from $X \sqcup X'$ to $Y \sqcup Y'$. We denote its equivalence class by $d = c \sqcup c'$ and call it the disjoint union. As the notation suggests, one can check that the equivalence class does not depend on the representatives of c and c'. In fact, by carefully going through the definitions, we see that

$$\sqcup : \mathbf{nCob} \times \mathbf{nCob} \longrightarrow \mathbf{nCob}.$$

is a bifunctor. We can also form disjoint union of any finite number of objects or morphisms. To simplify notation we assume left-bracketing for such unions.

There also exists a symmetry. Consider the natural diffeomorphism

$$X \sqcup Y \longrightarrow Y \sqcup X.$$

By the previous section this induces an isomorphism τ_{XY} and one easily sees that the relation

$$\tau_{XY}\tau_{YX} = \mathrm{id}_{X \sqcup Y}$$

holds.

To obtain more structure on the category, we consider $\varnothing$ an object of **nCob** and identify $X \sqcup \varnothing = \varnothing \sqcup X = X$. By abuse of notation we also consider $\varnothing$ an endomorphism of $\varnothing$ and get similar identifications for morphisms.

One can now check all the coherence relations required by a symmetric monoidal category, and indeed we have the following proposition.

Proposition 2. The triple $(\mathbf{nCob}, \sqcup, \varnothing, \tau)$ satifies the axioms of a symmetric monoidal category.

Decomposition into connected cobordisms. Every manifold is diffeomorphic to the disjoint union of its connected components. The question now arises if we can write any cobordism as the disjoint union of *connected cobordisms*, that is cobordisms with the underlying manifold connected. This is indeed possible.

Proposition 3. Any morphism in **nCob** can be written in the form

$$c = c_{\phi}\,(c_1 \sqcup \cdots \sqcup c_l)\,c_{\psi} \tag{1}$$

with each c_ν connected.

Proof. Let

$$c = [M, i, j] : X \longrightarrow Y$$

be a morphism and

$$M \cong M_1 \sqcup \cdots \sqcup M_l.$$

its composition into connected components. We set $X_\nu = i^{-1}(M_\nu)$ and denote the restriction of i to this manifold by

$$i_\nu : X_\nu \longrightarrow M_\nu.$$

Repeating this construction for Y and j we get cobordisms

$$c_\nu = [M_\nu, i_\nu, j_\nu] : X_\nu \longrightarrow Y_\nu.$$

Since the sets $X_1, \ldots, X_l$ are disjoint, there is a natural diffeomorphism

$$\phi : X \longrightarrow X_1 \sqcup \cdots \sqcup X_l$$

and similarly

$$\psi : Y_1 \sqcup \cdots \sqcup Y_l \longrightarrow Y.$$

These definitions clearly satisfy equation (1). □

Cobordisms in dimension two

Skeleton. We apply Proposition 1 and classify objects up to isomorphism in **2Cob**.

Definition. A *skeleton* of a category **C** is a full subcategory **S** such that every object in **C** is isomorphic to exactly one object in **S**.

We know that every oriented closed and connected one-manifold is diffeomorphic to $S^1 = \mathbf{R}/\mathbf{Z}$. Without loss of generality this diffeomorphism is orientation preserving with respect to the positive orientation induced by **R**. If necessary, we compose the diffeomorphism with the orientation reversing map $x \mapsto -x$. Denote the disjoint union of n spheres by $\mathbf{n} = S^1 \times \{1, 2, \ldots, n\}$ and the empty set by $\mathbf{0}$. Then every closed manifold is diffeomorphic, and by Proposition 1, isomorphic in **2Cob** to some **n**. This prooves the first part of the following proposition.

Proposition 4. The set $\mathbf{S} = \{\mathbf{n} \mid n = 0, 1, 2, \ldots\}$ forms a skeleton for **2Cob**.

It remains wo show that for $n \neq m$ the objects **n** and **m** are not isomorphic in **2Cob**.

Proof. Let $c : \mathbf{n} \to \mathbf{m}$ be a cobordism with manifold M and c^{-1}, with manifold N, its inverse. This means that gluing these manifolds along **m** gives a manifold diffeomorphic to $\mathbf{n} \times [0, 1]$. Consider the composition

$$f : \mathbf{m} \xrightarrow{i} M \cup_h N \xrightarrow{\psi} \mathbf{n} \times [0, 1] \longrightarrow \mathbf{n},$$

where the first map is the natural embedding onto the boundary where M and N are glued together, and the last map is projection to the first component. This map is coninuous. Let $X_1, \ldots, X_n$ be the connected components of **n**, then **m** is the disjoint union of the open sets $f^{-1}(X_1), \ldots, f^{-1}(X_n)$. We argue that neither of these sets is empty.

Let $x \in X_\nu$. The path $t \mapsto \psi^{-1}(x, t)$ goes from N to M in the glued manfifold. It thus has to cross the image of **m** somewhere. Pick $y \in \mathbf{m}$ such that $i(y) = \psi^{-1}(x, t)$ for some t. It follows that $f(y) = x$ and hence

$y \in f^{-1}(X_\nu)$. This shows that the number of connected components in $\mathbf{m}$ is at least n. Repeating the argument with $\mathbf{n}$ and $\mathbf{m}$ interchanged we conclude that $n = m$. $\square$

Disjoint union revisited. If we restrict the disjoint union to $\mathbf{S}$ we obtain a bi-functor

$$\sqcup : \mathbf{S} \times \mathbf{S} \longrightarrow \mathbf{2Cob}$$

This, however, does not give a monoidal structure on $\mathbf{S}$, since the image of $\sqcup$ is not contained in $\mathbf{S}$. We thus have to modify the disjoint union by composing it with the functor which sends every object in $\mathbf{nCob}$ to its representative in $\mathbf{S}$. For objects we obviously set

$$\mathbf{n} \sqcup_{\mathbf{S}} \mathbf{m} = (\mathbf{n} + \mathbf{m}).$$

For morphisms on the other hand we have to do some work.

Suppose that $c : \mathbf{n} \to \mathbf{m}$ and $c' : \mathbf{n}' \to \mathbf{m}'$ are morphisms in $\mathbf{S}$, then

$$c \sqcup c' : \mathbf{n} \sqcup \mathbf{n} \longrightarrow \mathbf{m} \sqcup \mathbf{m}.$$

To obtain a morphism

$$c \sqcup_{\mathbf{S}} c' : (\mathbf{n} + \mathbf{n}') \longrightarrow (\mathbf{m} + \mathbf{m}')$$

we have to precompose and compose $c \sqcup c'$ with morphisms $(\mathbf{n} + \mathbf{n}') \to \mathbf{n} \sqcup \mathbf{n}'$ and $\mathbf{m} \sqcup \mathbf{m} \to (\mathbf{m} + \mathbf{m}')$ respectively. For $n, n' \in \mathbf{N}$, let

$$\phi_{nn'} : \mathbf{n} \sqcup \mathbf{n}' \to (\mathbf{n} + \mathbf{n}') \tag{2}$$

be the diffeomorphism given by

$$\phi_{nn'}(x, \nu) = \begin{cases} (x, \nu) & \text{if } (x, \nu) \in \mathbf{n} \\ (x, \nu + n) & \text{if } (x, \nu) \in \mathbf{n}' \end{cases}$$

Set

$$c \sqcup_{\mathbf{S}} c' = c_{\phi_{nn'}} \left(c \sqcup c' \right) c^{-1}_{\phi_{mm'}}.$$

One can check that axioms of a monoidal category hold for the functor

$$\sqcup_{\mathbf{S}} : \mathbf{S} \times \mathbf{S} \longrightarrow \mathbf{S}.$$

In fact, the monoidal structure even is strict, since we have only one object per isomoprhism class.

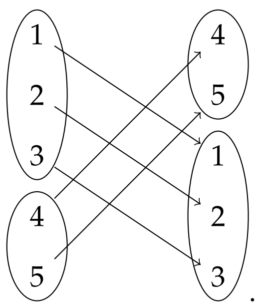
Symmetry revisited. We discuss symmetry in a similar fashion. Again, since we have only on object in each isomorphism class the symmetrie $\tau^{\mathbf{S}}_{\mathbf{kl}}$ should in fact be an automorphism of $(\mathbf{k}+\mathbf{l})$. We take the same route as in the previous paragraph and define the symmetry using the morphisms given by equation (2) and the commutative diagram

$$\begin{array}{ccc} (\mathbf{k}+\mathbf{l}) & \xrightarrow{\tau^{\mathbf{S}}_{\mathbf{kl}}} & (\mathbf{l}+\mathbf{k}) \\ {\scriptstyle \phi_{kl}}\uparrow & & \uparrow{\scriptstyle \phi_{lk}} \\ \mathbf{k}\sqcup\mathbf{l} & \xrightarrow{\tau_{\mathbf{kl}}} & \mathbf{l}\sqcup\mathbf{k}. \end{array}$$

To be more explicit we can check, that the symmetry on $(\mathbf{k}+\mathbf{l}) = \mathbf{n}$ is induced by the diffeomorphism σ_{kl} of $S^1 \times \{1, \dots, n\}$ with

$$\sigma_{kl}(x, \nu) = \begin{cases} (x, \nu + l) & \text{if } \nu \leq k \\ (x, \nu - k) & \text{if } \nu > k \end{cases} \tag{3}$$

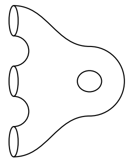
For $k = 2$ and $l = 3$ this can be visualised as



Classification of morphisms

As we have made explicit the symmetric monoidal category **S** and its relation to **2Cob**, we can work with the skeleton. From now on we write **2Cob** for the skeleton. After characterising the objects in **2Cob** we turn to morphisms. Silently drop any reference to the skeleton

Visual representation. In a first step we develop a system to represent cobordisms graphically. This allows us to derive some insight from visual intuition. Consider the following picture.



It clearly describes a manifold M, but not a cobordism, since it contains no information about the embeddings i and j into the boundary or orientation. Suppose we want to view the manifold as a cobordism from $\mathbf{n}$ to $\mathbf{m}$. Obvioulsy the number of boundary components of M must equal $n + m$.

We label the boundary components by $\bar{1}, \bar{2}, \ldots, \bar{n}$ and $1, 2, \ldots, m$ and interpret these as follows. Pick an orienation of M and orientation reversing diffeomorphisms

$$i_k : S^1 \to \overline{\Sigma}_k$$

where $\overline{\Sigma}_k$ denotes the boundary component labeled by $\bar{k}$. This induces an orientation reversing embedding

$$\begin{aligned} i : \qquad \mathbf{n} &\longrightarrow \partial M \\ (z, k) &\longmapsto i_k(z) \end{aligned}$$

Similarily, for each boundary component labeled k, we choose an orientation preserving diffeomorphism and obtain an orientation preserving embedding $j : \mathbf{m} \to \partial M$. Since we labeled all components, we clearly have $i(\mathbf{n}) \sqcup j(\mathbf{m}) = \partial M$. This finally specifies a cobordism.

We argue that the cobordism class is independent of both the choice of diffeomorphisms and orientation of M. Assume that i_k and i'_k are two such orientation reversing diffeomorphisms, then

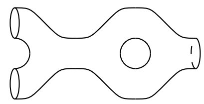
$$i_k^{-1} \circ i'_k : S^1 \longrightarrow S^1$$

is an orientation preserving diffeomorphism and thus isotopic to the identity. Hence $\phi = i^{-1} i'$ is isotopic to the identity on $\mathbf{n}$. Applying Proposition 1 we get

$$[M, i, j] = c_\phi [M, i, j] = [M, i\phi, j] = [M, i', j].$$

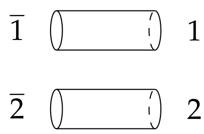
Repeating the argument for j shows that the cobordism in independent of the choice of diffeomorphisms.

As a convention we draw the source boundaries on the left-hand side an the target boundaries on the right hand side and label them consecutively from top to bottom. The picture from the beginnig would thus be drawn as follows.

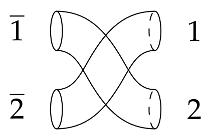


The Twist. We may ask ourselves if, given the source and target, the labeling is really necessary to distinguish non-equivalent cobordisms. As we will show later the labeling does not matter if the manifold is connected, but we can easily construct a non-connected example where it is essential.

Recall that the identity morphism on **2** is represented by the cobordism (M, i, j) with manifold $M = \mathbf{2} \times [0,1]$ and embeddings $i = \mathrm{id} \times 0$ and $j = \mathrm{id} \times 1$.



Define an orientation preserving diffeomorphism τ of $\mathbf{2} = S^1 \times \{0,1\}$ by $\tau(z,s) = (z, 1-s)$ for $s = 0,1$. This gives a cobordism (M, i, j') with $j' = \tau \times 1$. Its equivalence class T is represented by the following picture.



(T)

Suppose T and $\mathrm{id}_{\mathbf{2}}$ are equivalent. Then there exists an orientation preserving diffeomorphism g of M such that $gi = i$ and $gj = j'$. Since M is the union of the two connected components $M_0 = S^1 \times \{0\} \times [0,1]$ and $M_1 = S^1 \times \{1\} \times [0,1]$ the map g has to preserve these. Assume $g(M_0) = M_0$. Then

$$j'(S^1 \times \{0\}) = S^1 \times \{1\} \times \{1\} \subset M_1.$$

But on the other hand we have

$$gj(S^1 \times \{0\}) \subset g(M_0) = M_0$$

contradicting $gj = j'$. The same argument (using i and i' instead of j and j') also yields a contradiction for $g(M_0) = M_1$. Thus no such g can exists.

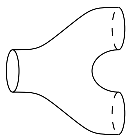
Observe that the second cobordism class T is induced by equation (3) and thus is the symmetry on $\mathbf{1} \sqcup \mathbf{1}$. We call this morphism the *twist*.

Generators. Consider a functor $F : \mathbf{C} \to \mathbf{D}$ and morphisms α and β in $\mathbf{C}$. If we know the functor's value on these morphisms, the functorial property allows us to deduce its value on the composition. Similarily, if the categories and the functor are assumed to be monoidal we have $F(\alpha \,\square\, \beta) = F(\alpha) \,\square\, F(\beta)$. This shows that we do not need all morphisms to completely determine a functor.

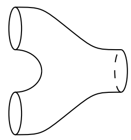
Definition. A subset G of the set of all morphisms of a symmetric monoidal category is said to *generate* that category if every morphism can be obtained by repeatedly composing and boxing (i.e. applying the monoidal functor) morphisms in G.

Here we require that, for any object, the identity needs to be generated by G, too. This has the advantage that, if we know the values of a symmetric monoidal functor on G, we can deduce its value on both objects and every other morphism.

Morphisms we have have seen so far are the twist T and the identity $I = \mathrm{id}_{\mathbf{1}}$. We give additional examples of simple cobordisms with manifold a disk or a disk with two holes.



(P)



$(\overline{P})$

(D)

$(\overline{D})$

Theorem 2. The set

$$\{I, T, D, P, \overline{D}, \overline{P}\} \tag{G}$$

generates $\mathbf{2Cob}$.

This theorem now enables us to describe any symmetric monoidal functor on $\mathbf{2Cob}$.

Corollary 3. A symmetric monoidal functor Z from $\mathbf{2Cob}$ to any symmetric monoidal category is completely determined by its value on the morphisms $D, P, \overline{D}$ and $\overline{P}$ and the object $Z(\mathbf{1})$.

Proof. As we haveremarked at the beginning of this section, the values of Z on the generating set of Theorem 2 completely determine the functor. It remains to calculate $Z(I)$ and $Z(T)$. Since functors preserve identities and

$I = \mathrm{id}_{\mathbf{1}}$ we have $Z(I) = \mathrm{id}_{Z(\mathbf{1})}$. Also, since Z is a symmetric functor, we have

$$Z(T) = Z(\tau_{\mathbf{1},\mathbf{1}}) = \tau'_{Z(1),Z(1)},$$

where τ' is the symmetry on the target category. □

Proof of Theorem 2. The proof relies on ideas from Morse theory and the notion of *elementary cobordism*. For an in-depth development of this topic we refer the reader to (Milnor 1965), especially Section 3. It is shown there, that every cobordism can be obtained by glueing together elementary cobordisms. Moreover, exactly one of the connected components of an elementary cobordisms is again an elementary cobordisms and the remaining are cylinders. It thus follows from Proposition 3 that the morphisms of the form c_ϕ for ϕ a diffeomorphism together with the elementary cobordisms generate **nCob**.

In dimension $n = 2$ one can show (Hirsch 1976, Theorem 9.3.4) that a connected elementary cobordism is equivalent to $D, P, \overline{D}$ or $\overline{P}$. It remains to classify the morphisms c_ϕ.

Let $\phi : \mathbf{n} \to \mathbf{n}$ be an orientation preserving diffeomorphism. For $k = 1, \dots, n$ the map ϕ restricts to

$$\phi : S^1 \times \{k\} \longrightarrow S^1 \times \{\sigma(k)\}$$

for some permuation σ. This induces an orientation preserving diffeomorphism on S^1 which is isotopic to the identiy. Hence ϕ is isotopic to

$$\tilde{\sigma} : (x, k) \longmapsto (x, \sigma(k)) .$$

The permutation group is generated by neighboring transpositions, so we can write $\tilde{\sigma} = \tilde{\sigma}_1 \cdots \tilde{\sigma}_m$ with $\sigma_\nu = (k_\nu, k_\nu + 1)$. Using the definition of the disjoint union one easily checks that

$$\tilde{\sigma}_\nu = \underbrace{I \sqcup \cdots \sqcup I}_{k_\nu - 1 \text{ times}} \sqcup T \sqcup \underbrace{I \sqcup \cdots \sqcup I}_{n - k_\nu - 1 \text{ times}} .$$

This completes the proof. □

In particular this also shows that there is a pacticular form to represent a morphism. That is for any morphism c there exists a finite number of morphisms

$$\{c_1, \dots, c_k\} \subset \{T, D, P, \overline{D}, \overline{P}\}$$

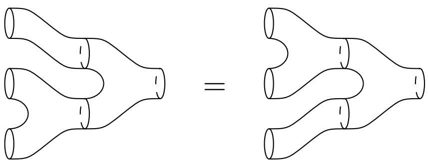
such that

$$c = \tilde{c}_1 \cdots \tilde{c}_k$$

where

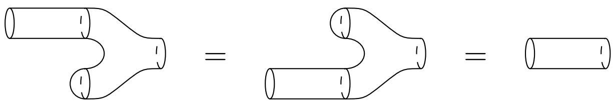
$$\tilde{c}_\nu = I \sqcup \cdots \sqcup I \sqcup c_\nu \sqcup I \sqcup \cdots \sqcup I.$$

Relations. In general, the representation of a morphism in terms of generators is not unique. We have, for example

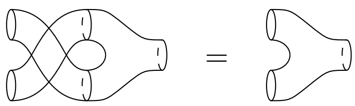


$$(I \sqcup P)P \quad = \quad (P \sqcup I)P \qquad (A')$$

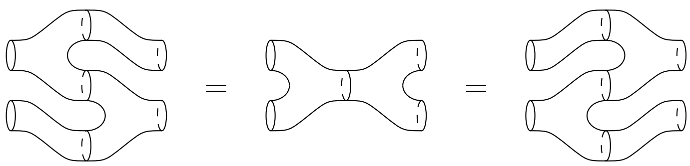
We call this kind of statement a *relation*. Using the visual representation we immediately see that the following additional relations hold in **2Cob**.



$$(I \sqcup D)P \quad = \quad (D \sqcup I)P \quad = \quad I \qquad (U')$$



$$TP \quad = \quad P \qquad (C')$$



$$(\overline{P} \sqcup I)(I \sqcup \overline{P}) \quad = \quad \overline{P}P \quad = \quad (I \sqcup \overline{P})(\overline{P} \sqcup I) \qquad (F')$$

Relations are simply another way of expressing commutative diagrams, with less emphasis on source and target of morphisms. Therefore, a comparison with the relations of section 1 prooves the following

Theorem 3. $(\mathbf{1}, P, D, \overline{P}, \overline{D})$ is a commutative Frobenius object in **2Cob**.

3 Classification of 2D-TFTs

With the machinery developed so far, we are now able to describe all symmetric monoidal functors $Z : \mathbf{2Cob} \to \mathbf{Vect}$. We call such functors *Two-Dimensional Topological Field Theories* or **2TFT** for short. Since symmetric monodidal functors preserve commutative Frobenius object we see that $A = Z(\mathbf{1})$ is a commutative Frobenius algebra. We denote this assignment by

$$\begin{aligned} F : \mathbf{2TFT} &\longrightarrow \mathbf{cFrob} \\ Z &\longmapsto (Z(\mathbf{1}), Z(P), \ldots)\,. \end{aligned} \tag{4}$$

Of course the question arises, wheter this map is one-to-one and onto and which kind of structure it preserves. It follows from **??** that F in indeed injective.

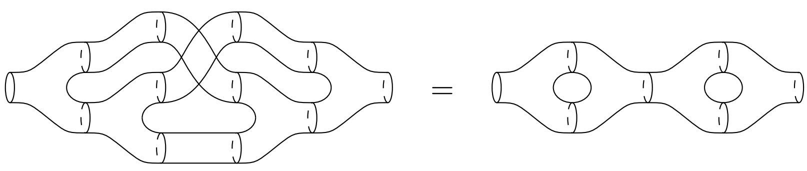
Completeness of Relations. To answer the second question we have to introduce the concept of completeness of relations.

Definition. A set of relations R of morphisms of a category is called *complete* if any relation between morphisms can be reduced to a trivial relation by repeatedly applying relations of R.

As shown in (Kock 2003, pp. 73) we have the following theorem for **2Cob**.

Theorem 4. The relations (A') through (F') together with the relations obtained from the symmetry τ are complete.

Now suppose (A, α, f, β, g) is a commutative Frobenius algebra. Set $Z(\mathbf{1}) = A$, $Z(P) = \alpha$, etc. We want to extend Z to a symmetric monoidal functor. This requires $Z(I) = \mathrm{id}_A$ and $Z(T) = \tau_{AA}$, where τ is the tensor symmetry. Z is now defined on the generating set (G) of **2Cob**. So for every morphism c in **2Cob** we may choose a representation in terms of generators and define $Z(c)$ be applying Z to every generator. Since the representation is not unique, this assignment is a priori not well-defined. For example consider the morphism c represented by.



We obtain this relation be applying (A') on the right and then (C') and (F'). But by assumption that A is a Frobenius algebra, all relations also hold in **Vect**, thus both definition of Z coincide. By the previous theorem we can apply this procedure to every representation. Hence Z is a well-defined symmetric monoidal functor.

Functorial Equivalence. We provide a slight overview over additional properties of the map F. For furhter details and proofs the reader is refered to (Abrams 1996, Section 5).

As the notation in (4) suggests we can make an even stronger statements about this map then it being bijective. The set of functors **2TFT** is, in fact, a category with morphisms being natural transformations that preserve the symmetric monoidal structure. For general symmetric monoidal categories this preservation requirement has to be formulated by coherence diagram. In our case, since we use a very simple skeleton, it reduces to the fact a natural transformation $\tau : Z \to Z'$ has to satisfy

$$\tau_{\mathbf{n}} = \tau_{\mathbf{1}}^{\otimes n}$$

and

$$\tau_{\mathbf{0}} = \mathrm{id}_{\mathbf{k}}.$$

The category **cFrob** has as objects all Frobenius algebras over the field **k**. A morphism between (A, α, f, β, g) and $(A', \alpha', f', \beta', g')$ is a **k**-linear isomorphisms $\phi : A \to A'$ sending the the structure maps α, f, β, g to their counterparts α', f', β', g', that is

$$\begin{aligned} \beta \circ \phi^{\otimes 2} &= \phi \circ \beta \\ f \circ \phi &= \phi^{\otimes 0} f' = f' \\ &\text{etc.} \end{aligned}$$

We are now able to define F on morphisms, sending $\tau : Z \to Z'$ to

$$F(\tau) = \tau_{\mathbf{1}} : F(Z) \longrightarrow F(Z')$$

where $F(Z) = Z(\mathbf{1})$ and $F(Z') = Z'(\mathbf{1})$ As detailed in (Abrams 1996, Theorem 3) this is an equivalence of categories.

Tensor products and direct sum. Since we have vector spaces underlying the definitions of **2TFT** and **cFrob** it is also natural to equip the two categories with direct sums and tensor products, or, in categorical terms, with a symmetric monoidal and additive structure. In **cFrob** we have to check that we can build new structure maps on the tensor product or direct sum from the maps on the factors or summands respectively, that also satisfy the axioms of a Frobenius algebra. For **2TFT** we define the tensor product and direct sum pointwise and then assert that this also gives a symmetric monoidal functor.

As a summary, we repeat the statement of Theorem 3 in (Abrams 1996)

Theorem 5. The functor F is an equivalence of categories which preserves tensor products and direct sums.

References

Abrams, Lowell (1996). "Two-dimensional topological quantum Field theories and Frobenius algebras". *Journal of Knot Theory and its Ramifications* 5, pp. 569–587.

Baez, John C. and James Dolan (Mar. 1996). "Higher-dimensional Algebra and Topological Quantum Field Theory". *J. Math. Phys.* 36.11, pp. 6073–6105.

Hirsch, Morris W. (1976). *Differential Topology*. Graduate Texts in Mathematics 33. Springer.

Kock, Joachim (2003). *Frobenius Algebras and 2D Topological Quantum Field Theories*. London Mathematical Society Student Texts 59. Cambridge University Press.

Mac Lane, Saunders (1971). *Categories for the Working Mathematician*. Graduate Texts in Mathematics 5. Springer.

Milnor, John Willard (1965). *Lectures on the H-Cobordism Theorem*. Mathematical notes. Princeton University Press.