HOMOTOPICAL AND HIGHER ALGEBRA – SEMINAR

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Week 4 – Extending down TFTs

M closed oriented n-mfld. Consider it as

$$\begin{array}{c} M \in Mor(\emptyset,\emptyset) \\ \downarrow Z \\ Z(M): \underset{\cong k}{Z(\emptyset)} \longrightarrow \underset{\cong k}{Z(\emptyset)} \in Hom(k,k) \cong k \end{array}$$

$\Leftrightarrow$ Z assigns to M an element of k, a diffeo. invariant.

NEW POINT OF VIEW: these diffeo. invariants are our new main object of interest.

Since we can also evaluate Z on n-mfld with boundary and on $(n-1)$-mfld without boundary, we try to compute $Z(M)$ by breaking M into pieces.

Ex. Σ_g: closed oriented 2-mfld of genus $g \geq 0$

Recall $\mathcal{C}at$(comm. Frobenius algebras) $\cong$ 2TFT

Given: a Frob.alg A with basis $\{T_0, T_1, \ldots, T_r\}$ such that $T_0 = 1_A$

multiplication $\mu: A \otimes A \longrightarrow A$

defined by $\mu(T_i \otimes T_j) = \mu_{ij}^k T_k$

Frobenius form $\varepsilon: A \longrightarrow k$

We define an associative nondegenerate pairing $\beta: A \otimes A \longrightarrow k$

$$(x \otimes y) \longmapsto \varepsilon(\mu(x \otimes y))$$

by $\beta(T_i \otimes T_j) = \beta_{ij}$

nondegeneracy of $\beta \Rightarrow \exists \gamma: k \longrightarrow A \otimes A$

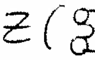
defined by $\gamma(1) = \gamma^{ij} T_i \otimes T_j$

such that $\beta_{ij}\gamma^{jk} = \iota_i^k$

$\gamma^{ij}\beta_{jk} = \iota_k^i$ Kronecker delta

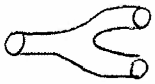
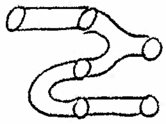
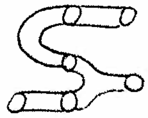
i.e. $((\beta_{ij})_{i,j=1}^n)^{-1} = (\gamma^{ij})_{i,j=1}^n$

We've seen that given a Frobalg. we obtain a 2TFT by setting

$Z(S^1) = A$, $Z($$) = \mu$, $Z($$) = \varepsilon$, $Z($$) = \beta$, $Z($$) = \gamma$

and $Z($$) = \begin{array}{l} k \longrightarrow A \\ 1_k \mapsto 1_A \end{array}$

We also want to define a comultiplication $\delta : A \to A \otimes A$ as

 $:=$  $=$ 

Hence we define $\delta(T_k) = \delta_k^{ij} T_i \otimes T_j$

by $\delta_k^{ij} = \gamma^{ie} \mu_{ek}^{j} = \mu_{kf}^{i} \gamma^{fj}$

and set $Z($$) = \delta$.

CONCRETE EX. $\mathbb{C}$ as vect. space over $\mathbb{R}$ with basis $T_0 = 1$, $T_1 = i$

usual multiplication

$$\varepsilon : \mathbb{C} \longrightarrow \mathbb{R}, \quad z \longmapsto \mathrm{Re}(z)$$

$$\Updownarrow$$

$$\beta : \mathbb{C} \otimes \mathbb{C} \longrightarrow \mathbb{R}, \quad z_1 \otimes z_2 \longmapsto x_1 x_2 - y_1 y_2$$

$\big(\mathrm{Re}(z_i) = x_i,\ \mathrm{Im}(z_i) = y_i\big)$

computations... $\Longrightarrow$

$$\mu_{00}^{0} = 1,\ \mu_{00}^{1} = 0,\ \mu_{01}^{0} = 0,\ \mu_{01}^{1} = 1$$

$$\mu_{10}^{0} = 0,\ \mu_{10}^{1} = 1,\ \mu_{11}^{0} = -1,\ \mu_{11}^{1} = 0$$

$$\beta_{00} = 1,\ \beta_{10} = 0,\ \beta_{01} = 0,\ \beta_{11} = -1$$

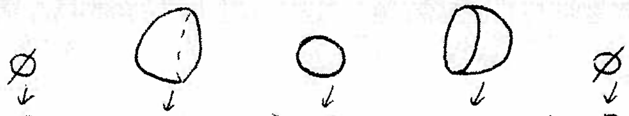
$$(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\delta : \mathbb{C} \longrightarrow \mathbb{C} \otimes \mathbb{C}$$

$$T_0 \longmapsto T_0 \otimes T_0 - T_1 \otimes T_1$$

$$T_1 \longmapsto T_0 \otimes T_1 + T_1 \otimes T_0$$

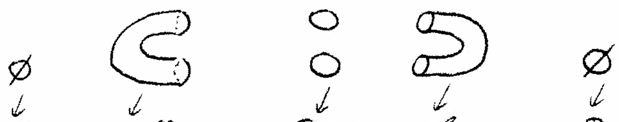
$\Sigma_0 = S^2$:



$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{R} \\ 1 & \longmapsto & 1 & \longmapsto & 1 \end{array}$$

$Z(S^2) = 1 \in \mathbb{R}$

$\Sigma_1 = \mathbb{T}^2$



$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\gamma} & \mathbb{C}\otimes\mathbb{C} & \xrightarrow{\beta} & \mathbb{R} \\ 1 & \longmapsto & T_0\otimes T_0 - T_1\otimes T_1 & \longmapsto & 2 \end{array}$$

$Z(\Sigma_1) = 2$

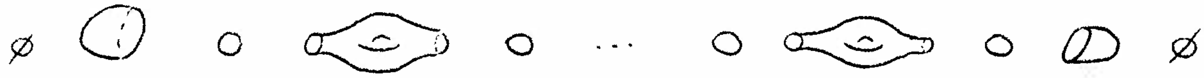
$\left(\text{in fact for any Frob. alg } \quad 1 \xmapsto{\gamma} \gamma^{ij}\, T_i \otimes T_j \xmapsto{\beta} \beta_{ij}\gamma^{ij} = n = \dim(A)\right)$

Now consider

:

$$\begin{array}{ccccc} \mathbb{C} & \xrightarrow{\delta} & \mathbb{C}\otimes\mathbb{C} & \xrightarrow{\mu} & \mathbb{C} \\ T_0 & \longmapsto & T_0\otimes T_0 - T_1\otimes T_1 & \longmapsto & 2\cdot T_0 \\ T_1 & & \longmapsto & & 2\cdot T_1 \end{array}$$

Hence, Σ_g :



$$\begin{array}{ccccccccccc} \mathbb{R} & \longrightarrow & \mathbb{C} & \xrightarrow{\cdot 2} & \mathbb{C} & \xrightarrow{\cdots} & \mathbb{C} & \xrightarrow{\cdot 2} & \mathbb{C} & \xrightarrow{\mathrm{Re}} & \mathbb{R} \\ 1 & \longmapsto & 1 & \longmapsto & 2 & \longmapsto \cdots \longrightarrow & 2^{g-1} & \longmapsto & 2^g & \longrightarrow & 2^g. \end{array}$$

$\Longrightarrow \quad Z(\Sigma_g) = 2^g \in \mathbb{R}.$

We try to go on with this process for $n>2$.

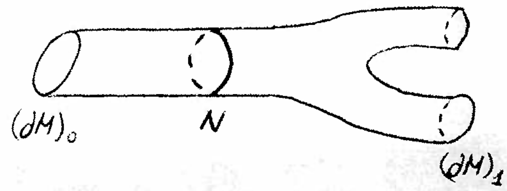
M oriented n-mfld, consider it as $M \in Mor(\emptyset, \partial M)$

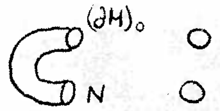
$$\begin{array}{l} \downarrow Z \\ Z(M) : \underset{\cong k}{Z(\emptyset)} \longrightarrow Z(\partial M) \end{array}$$

we interpret it as $Z(M) \in Z(\partial M)$ ↑ vector space.

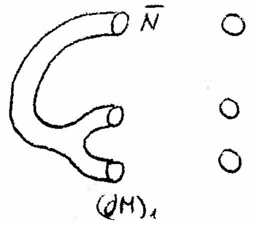
In order to translate functoriality:

take a closed submfld N of codim 1

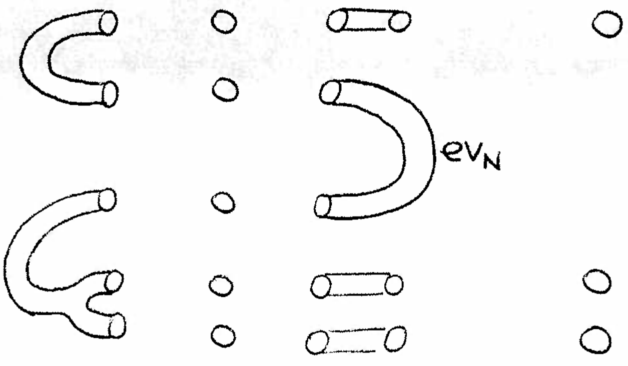
M: 

M_0: $\emptyset$ 

$$k \longrightarrow \underset{\substack{\cup\!\!\!\shortmid \\ Z(M_0)}}{Z((\partial M)_0) \otimes Z(N)}$$

M_1: $\emptyset$ 

$$k \longrightarrow \underset{\substack{\cup\!\!\!\shortmid \\ Z(M_1)}}{Z(\bar{N}) \otimes Z((\partial M)_1)}$$

M: $\emptyset$ 

$$k \longrightarrow \begin{array}{c} Z((\partial M)_0) \otimes Z(N) \otimes \\ \otimes Z(\bar{N}) \otimes Z((\partial M)_1) \\ \cong \\ Z(\partial M) \otimes Z(N) \otimes Z(\bar{N}) \end{array} \longrightarrow Z(\partial M)$$

$$1 \longmapsto Z(M_0) \otimes Z(M_1) \longmapsto Z(M).$$

$\Longrightarrow$ We have found a rule for computing $Z(M)$ given a decomposition $M = M_0 \amalg_N M_1$

if N is closed and of codim 1.

We've seen that for $n=2$ is enough (only needed pieces )

Can we find $\forall n$ a list of "simple" n-mfld with boundary such that any n-mfld is obtained by gluing these?

For $n \geq 3$ this gets complicated:
- we don't have a complete classification of n-mflds
- if $n \gg 0$, ~~that~~ this process doesn't help us a lot, since we don't know how the submflds of codim 1 look like.

We could use triangulations, but the gluing gets more complicated (we have "corners")

$\to$ We will want to allow us to glue along submanifolds which have boundary themselves.

FIRST ATTEMPT TO EXTEND THE DEFINITION OF A nTFT:

We need these data:

(1) M^n closed oriented $\Longrightarrow Z(M) \in k$

(2) N^{n-1} closed oriented $\Longrightarrow$ a k-vect. space $Z(N)$, if $N = \emptyset$: $Z(N) = Z(\emptyset) \cong k$ canonically

(3) M^n oriented $\Longrightarrow$ an element $Z(M) \in Z(\partial M)$, if $\partial M = \emptyset$ $Z(M)$ should correspond to the element given in (1) under the $\cong$ from (2).

(4) P^{n-2} closed oriented $\Longrightarrow Z(P)$ a k-linear category, i.e. $Z(P)$ a category st. $\forall X, Y \in Obj(Z(P))$ $Mor_{Z(P)}(X,Y)$ is a k-vect. space and $Mor_{Z(P)}(X,Y) \times Mor_{Z(P)}(Y,Z) \longrightarrow Mor_{Z(P)}(X,Z)$ is bilinear, if $P = \emptyset$ then $Z(P) = Z(\emptyset) = \mathcal{V}ect(k)$.

5) N^{n-1} oriented $\Longrightarrow$ an object $Z(N) \in Obj(Z(\partial N))$, if $\partial N = \emptyset$: $Z(N)$ should correspond to the vect. space $Z(N) \in Obj(\mathcal{V}ect(k))$ given in (2).

This first attempt is not enough we have some

PROBLEMS:
- $\dim(B) = n$, B a bordism

  we didn't specify a method for defining a linear map $Z(B): Z(M) \to Z(N)$ $\forall$ $M \neq \emptyset$
- $\dim(B) = n-1$, B a bordism

  we didn't specify how to define the functor $Z(B): Z(M) \to Z(N)$ when $M \neq \emptyset$.
- we don't have any condition for gluings, nor for disjoint unions.

We try to refine the definition.

A STRICT 2-CATEGORY $\mathcal{C}$ consists of:

- a collection of objects $X, Y, Z, \dots$
- $\forall X, Y$ objects, $Map_{\mathcal{C}}(X,Y)$ is a category
- $\forall X$ object, a distinct object $id_X \in Map_{\mathcal{C}}(X,X)$
- $\forall X, Y, Z$ objects, a composition functor

$$\Psi: Map_{\mathcal{C}}(X,Y) \times Map_{\mathcal{C}}(Y,Z) \longrightarrow Map_{\mathcal{C}}(X,Z)$$

$$\Psi(F \circ F', G \circ G') = \Psi(F,G) \circ \Psi(F',G')$$

- $$\Psi(F, id_Y) = F$$
  $$\Psi(id_X, G) = G$$

- $$\begin{tikzcd}
Map_{\mathcal{C}}(W,X) \times Map_{\mathcal{C}}(X,Y) \times Map_{\mathcal{C}}(Y,Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Map_{\mathcal{C}}(W,Y) \times Map_{\mathcal{C}}(Y,Z) \arrow[d] \\
Map_{\mathcal{C}}(W,X) \times Map_{\mathcal{C}}(X,Z) \arrow[r] & Map_{\mathcal{C}}(W,Z)
\end{tikzcd}$$

This is also called a CATEGORY $\underbrace{\text{ENRICHED IN CATEGORIES}}$.

this means that $\forall X, Y$ objects $Map_{\mathcal{C}}(X,Y)$ is a [redacted] category and satisfy the listed conditions

Remark In fact we can enrich a category $\mathcal{C}$ in any monoidal category $\mathcal{M}$ by requiring that $Map_{\mathcal{C}}(X,Y)$ is an object in $\mathcal{M}$. For example: ordinary categories are enriched in **Sets**, k-linear categories are enriched in $\mathrm{Vect}(k)$.

Example $\mathrm{Vect}_2(k)$

- OBJECTS cocomplete k-linear categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$

  i.e. $\mathcal{A}$ with objects $X, Y, Z, \dots$ is such that

  $$\text{k-linear} \begin{cases} \cdot\ \mathrm{Hom}_{\mathcal{A}}(X,Y) \text{ is } k\text{-vect. space } \forall X,Y \\ \cdot\ \mathrm{Hom}_{\mathcal{A}}(X,Y) \times \mathrm{Hom}_{\mathcal{A}}(Y,Z) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(X,Z) \text{ is bilinear} \end{cases}$$

  $$\text{cocomplete} \begin{cases} \cdot\ \text{closed under } \oplus \\ \cdot\ \text{closed under coker}(-) \end{cases}$$

  $\Updownarrow$ all colimits exist

- $\mathrm{Map}_{\mathrm{Vect}_2(k)}(\mathcal{A}, \mathcal{B})$ is the category of cocontinuous k-linear ~~functors~~ functors

  (cocontinuous $\Longleftrightarrow$ commutes with colimits)

- $id_{\mathcal{A}}$ the identity functor
- composition clear

Example $\mathrm{Cob}_2(n)$ NOT A STRICT 2-CATEGORY

we try to define it:

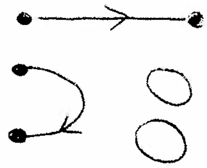
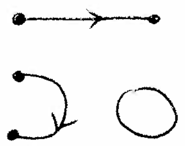
- OBJECTS closed oriented $(n-2)$-manifolds $M, N, P, \dots$
- $\mathrm{Map}_{\mathrm{Cob}_2(n)}(M,N)$ category with
  - OBJECTS bordisms B from M to N

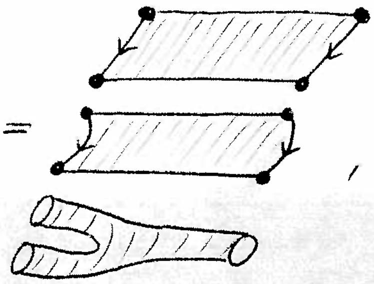
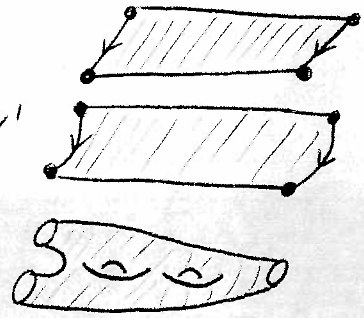
    i.e. B^{n-1} mfld together with diffeo $\partial B \cong \overline{M} \amalg N$

  - MORPHISMS $\forall B, B' \in \mathrm{Obj}(\mathrm{Map}_{\mathrm{Cob}_2(n)}(M,N))$

    $\mathrm{Hom}(B,B') = \{$ oriented diffeo. classes of oriented bordisms from B to B' such that the diffeo.s are trivial on the common boundary $\partial B \cong \overline{M} \amalg N = \partial B' \}$

Picture/Ex. (i) $M = \begin{matrix} + \bullet \\ + \bullet \\ - \bullet \end{matrix}$ $N = \bullet +$

(ii) $B =$  , $B' =$  $\in Obj(Map(M,N$

(iii) $X =$  , X' 

$$X, X' \in Hom(B, B') \quad , X \not\cong X'$$

Rem. $\partial X = \partial X' = \overline{B'} \amalg_{M \amalg \overline{N}} ((M \amalg \overline{N}) \times [0,1]) \amalg_{\overline{M} \amalg N} B'$

As pointed out $Cob_2(n)$ is not a strict 2-category. We encounter some difficulties:

(1) given $B: M \to M'$ and $B': M' \to M''$ we want to define a composition by gluing along M'. We need a well defined composition, not up to diffeo.
$\Rightarrow$ we need to define a smooth structure on the composition (for ex. fix a smooth collar nbhd of M' in B and B').

(2) given B, B' and $B'': M'' \to M'''$ we want
$$(B \amalg_{M'} B') \amalg_{M''} B'' \overset{!}{=} B \amalg_{M'} (B' \amalg_{M''} B'')$$
we can arrange to have a diffeo, but equality is difficult.

We will change this definition in order to fix these problems.

For the moment, suppose we have a good definition that takes care of these problems.

<u>Def.</u> A 2-EXTENDED TOPOLOGICAL FIELD THEORY of dim. n is a symmetric monoidal functor

$$Z: Cob_2(n) \longrightarrow Vect_2(k)$$

between 2-categories.

$\hookrightarrow$ we need to understand $Cob_2(n)$ and $Vect_2(k)$ as <u>symmetric monoidal</u> 2-categories.

For $Cob_2(n)$ it's easy $\quad \otimes := \amalg$

$$\mathbb{1}_{Cob_2(n)} = \emptyset^{(n-2)}$$

For $Vect_2(k)$ it's more complicated. We can define a $\otimes$ between categories and then $\mathbb{1}_{Vect_2(k)} = Vect(k)$

We called this new functor extended, because in fact it incorporates our old definition of nTFT:

For $\mathcal{C}$ a symm. mon. 2-category

$$\Longrightarrow \Omega\mathcal{C} := Map_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}) \text{ is a symm. mon. category}$$

We see that
$$\Omega\, Cob_2(n) = Map_{Cob_2(n)}(\emptyset^{n-2}, \emptyset^{n-2}) \cong Cob(n)$$
$$\Omega\, Vect_2(k) = Map_{Vect_2(k)}(Vect(k), Vect(k)) \cong Vect(k).$$

Notice that ΩZ allows us to make computations by cutting along closed submfld of codim 1, Z, on the other hand, allows us to cut along submflds of codim 1 with boundary.

We would like to go on with this process, and cut N along a submfld P, $N = N_0 \amalg_P N_1$, but this makes sense in our definition only if P is closed.

So, the idea is to extend TFT's even more. We start with a definition.

Def. For $n \geq 0$, a STRICT n-CATEGORY is defined by induction on n:

$\underline{n=0}$ a strict 0-category is a set

$\underline{n>0}$ a strict n-category is a category enriched over strict $(n-1)$-categories

i.o.w: we have the following
1. a collection of objects $X, Y, Z, \ldots$
2. $\forall X, Y$ objects $Map(X,Y)$ is a strict $(n-1)$-category
3. $id_X \in Map(X,X)$ identity objects
   $Map(X,Y) \times Map(Y,Z) \to Map(X,Z)$ composition
   satisfying unity and associativity conditions.

Notice
- $\underline{n=1}$ we have a category enriched over set, i.e. a usual category
- $\underline{n=2}$ the definition of strict 2-category given above.

We will weaken this definition in order to have associativity up to isomorphism for the composition, and not strict equality.