Bicategories

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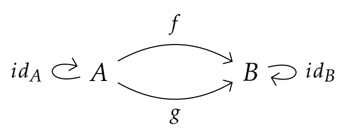
Abstract

This paper presents a brief introduction to the theory of bicategories, invented by Bénabou [6]. It was prepared for a seminar on Homotopical and Higher Algebra at ETH Zurich focusing on the bicategory 2Cob^{ext}. Note that certain parts are directly copied from [2].

1 What is a bicategory ?

Let's recall first what a category $\mathcal{C}$ is. At first sight, it is something like this :

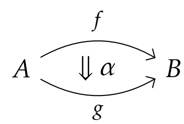


More precisely, we have a class of vertices called *the objects* of $\mathcal{C}$ and a class of edges that we call *arrows* or *morphims* of $\mathcal{C}$. Here $\text{Ob}(\mathcal{C}) = \{A, B\}$, $\text{Hom}(A,A) = \{id_A\}$, $\text{Hom}(A,B) = \{f,g\}$ and $\text{Hom}(B,B) = \{id_B\}$.

Additionally, in a category one is allowed to *compose* arrows in an associative and unital way :

- for every X, Y and Z in $\text{Ob}(\mathcal{C})$, there is an operation
$$\circ : \text{Hom}(X,Y) \times \text{Hom}(Y,Z) \to \text{Hom}(X,Z)$$
that satisfies
$$f \circ (g \circ h) = (f \circ g) \circ h$$
for every triple of composable morphisms f, g and h ;
- for every X in $\text{Ob}(\mathcal{C})$, there is a special arrow in $\text{Hom}(X,X)$ noted id_X which is a unit for the composition law, i.e. $\forall X, Y \in \text{Ob}(\mathcal{C}), \forall f \in \text{Hom}(X,Y)$ we have
$$f \circ id_X = f = id_Y \circ f.$$

Similarly this is how a part of a bicategory looks like :



A bicategory is thus obtained by adding some thicker arrows between usual arrows, some faces between edges. These are called *2-arrows* or *2-morphisms*. There will be a rule to compose two such arrows and by this we have some new layer of structure.

The last thing we need to know is the axioms that the composition rules should satisfy. Here is a delicate point ; we have two possibilities. The first one is to consider that the 1-arrows and the 2-arrows live their lives separately. We would then have the same axioms as for a category at each level. Or we could introduce 2-arrows in the axioms of the composition of 1-arrows in the following way : say that $\alpha \circ (\beta \circ \gamma)$ is equivalent to $(\alpha \circ \beta) \circ \gamma$ not only if there is an equality between the two but also if there is an invertible 2-morphism between them. By doing this, we *weaken* the categorical structure. We then can say that things commutes "up to an invertible 2-morphism"[1].

A category with the first (resp. second) structure will be called a *2-category* (resp. *bicategory*).

Definition 1.1 *A bicategory $\mathcal{B}$ consists of the following data,*

(1) a collection $\mathrm{Ob}(\mathcal{B})$ *(or $\mathcal{B}_0$) whose elements are called objects ;*

(2) categories $\mathcal{B}(a,b)$ for each pair of objects $a, b \in \mathcal{B}$. The objects of $\mathcal{B}(a,b)$ are the 1-morphisms while the morphisms of $\mathcal{B}(a,b)$ are the 2-morphisms. The composition law in $\mathcal{B}(a,b)$ is called vertical composition. We may also write $k\mathrm{Hom}(X,Y)$ for the class of all $k-arrows$ between two $k-1$-arrows X and Y ;

(3) functors :

$$c_{abc} : \quad \begin{array}{c} \mathcal{B}(b,c) \times \mathcal{B}(a,b) \to \mathcal{B}(a,c) \\ (g,f) \mapsto g \circ f \\ (\beta,\alpha) \mapsto \beta * \alpha \end{array} \quad ;$$

$$I_a : \mathbf{1} \to \mathcal{B}(a,a) \quad ;$$

for all objects $a, b, c \in \mathcal{B}$, 1-arrows f, g and 2-arrows α, β, where **1** *denotes the singleton category (thus the functor I_a is equivalent to specifying an identity element I_a. The functors c are called the horizontal compositions.*

[1] Exactly as topologists say : "up to homotopy".

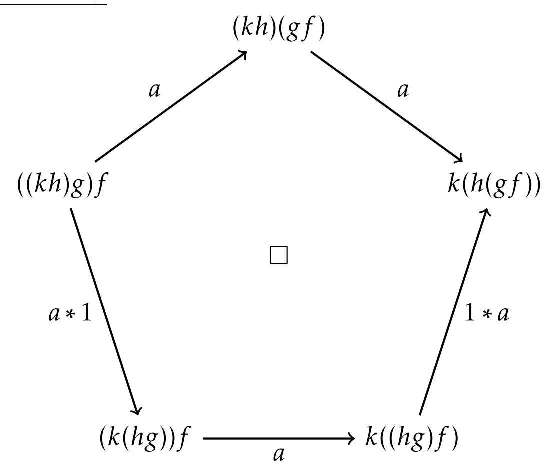
(4) Natural isomorphisms :

$$a : c_{abd} \circ (c_{bcd} \times id) \to c_{acd} \circ (id \times c_{abc})$$
$$\ell : c_{abb} \circ (I_b \times id) \to id$$
$$r : c_{aab} \circ (id \times I_a) \to id$$

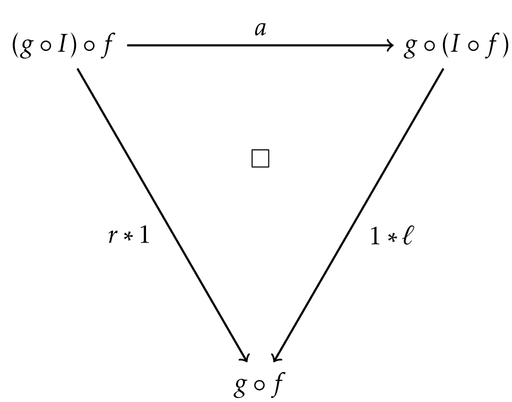
known, respectively, as the associator and left and right unitors. Thus invertible 2-morphisms $a_{h,g,f} : (h \circ g) \circ g \to h \circ (g \circ f)$, $\ell_f : I_b \circ f \to f$ *and* $r_f : f \circ I_a \to f$.

These are required to obey the following axioms :

The Pentagon Identity

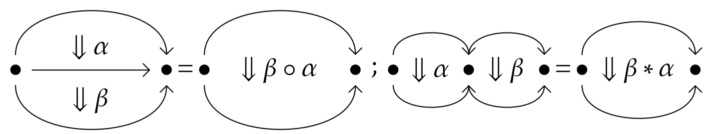


The Triangle Identity



Remark *As we shall see later, these two axioms ensure that results of computation don't depend on bracketings. Example :* $(fg)((hi)j)$ *is equivalent to* $f(g(h(ij)))$.

Achtung ! The horizontal composition of 1-morphisms is denoted $f \circ g$, which is contradictory with the notation for 2-morphisms. The vertical composition of 2-morphisms is denoted $\alpha \circ \beta$, while horizontal composition is denoted $\alpha * \beta$. These are standard notations.



2 Examples

2.1 Cat

The first example of a bicategory inside the theory of categories itself is Cat, the category of categories. A word of caution : we need to define Cat as the category of *small categories* i.e categories in which the class of objects and the class of morphisms are sets and not just classes. This is because in the realm of category theory, "Size does matter !"[1]

Let's describe Cat :

- Ob(Cat) = the class of all small categories ;
- $\mathrm{Hom}(X,Y) = \mathrm{Fun}(X,Y)$, the class of functors between two small categories X and Y ;
- $2\mathrm{Hom}(F,G) = \mathrm{Nat}(F,G)$ the class of natural transformation between two functors F and G.

Given with the usual rules concerning natural transformations and functors, we see that Cat is actually a 2-category, thus a bicategory.

2.2 Set with relations

The category Set has for objects the class of all sets and for morphisms, the functions between sets i.e specific subspaces of the cartesian product of two spaces.

Instead, consider the following structure :

- Ob(Set) is the class of all sets ;
- $\mathrm{Hom}(X,Y)$ is the set of all *relations* between X and Y, that is all the subsets $\Gamma \subset X \times Y$.

- 2Hom(Γ,Θ) is a set with one element if there is an *implication* $\Gamma \Rightarrow \Theta$ i.e if $\Gamma \subset \Theta$; or the empty set otherwise.

Let X, Y, Z be sets and $\Gamma \in \mathrm{Hom}(X,Y)$, $\Theta \in \mathrm{Hom}(Y,Z)$. Then the composition $\Theta \circ \Gamma \in \mathrm{Hom}(X,Z)$ is given by the subset $\{(x,z) \in X \times Z / \exists y \in Y, (x,y) \in \Gamma, (y,z) \in \Theta\}$. The identity morphism is $id_X = X \times X$.

For 2-morphisms, the composition is given by transitivity of implications.

The two operations are associative and make Set become a 2-category.

2.3 Alg with bimodules

This is an example of a bicategory that is *not* a 2-category :

- Ob(Alg) is the class of all algebras ;
- Hom(A,B) is the class of all $A-B$-bimodules, that is the class of all abelian groups on which A acts on the left and B acts on the right ;
- 2Hom(M,N) is the class of all bimodule homomorphisms between two bimodules M and N.

The composition of $M \in \mathrm{Hom}(A,B)$ and $N \in \mathrm{Hom}(B,C)$ is given by $N \circ M = M \otimes_B N$. This operation is *not* associative, indeed if P is in Hom(C,D), we only have an isomorphism $\Psi : M \otimes_B (N \otimes_C P) \to (M \otimes_B N) \otimes_C P$ i.e it is associative up to an invertible 2-arrow.

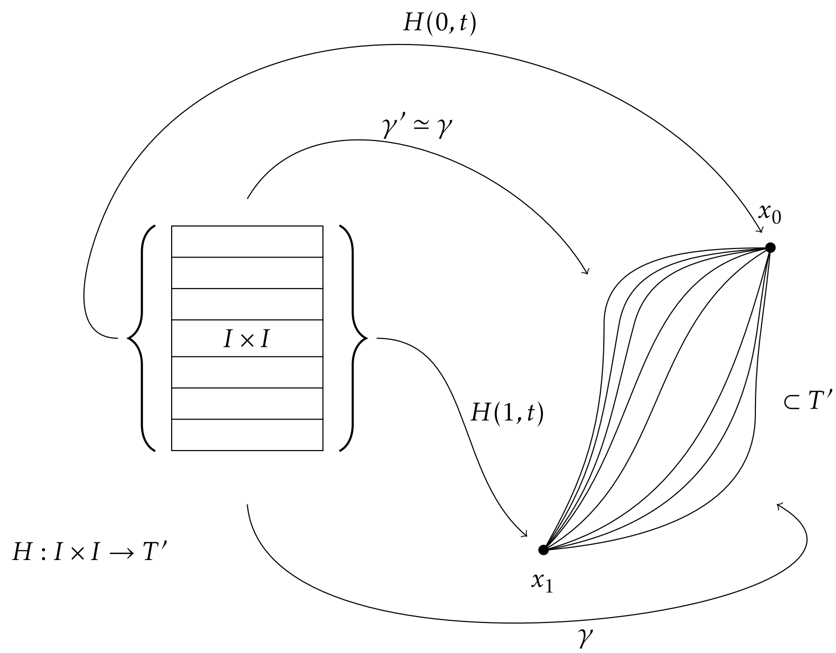
The reader is encouraged to verify that the Pentagon and Triangle Identities hold.

2.4 Π_2 : Top → 2Grpd

A crucial application of the theory of bicategories – and more generaly, higher categories – is the study of *homotopy types* of topological spaces. After recalling the definition of a homotopy, we will show how to build a bicategory from a topological space. This means that we *categorify* the notion of homotopy type (you can have a look at [5] for an introduction to homotopy theory).

Definition 2.1 *Let f and g be continuous maps between two topological spaces T and T'. We say that f is homotopic to g if there is a continuous map $H : [0,1] \times T \to T'$ such that $H(0,t) = f(t)$ and $H(1,t) = g(t)$ for all t in T.*

The principal property of a homotopy is that it itself is a continuous functions between topological spaces, so you can iterate the process by considering a homotopy between two homotopies and so on.



Let T be a topological space, then $\Pi_2(T)$ is the bicategory where

- the objects are the points of T ;
- the arrows are the continuons paths from a point to another ;
- the 2-arrows are homotopy classes of homotopies between paths.

The composition of homotopies is given by gluing two intervals and then rescaling. Such a transformation is homotopic to the identity, giving birth to the bicategorical structure.

The bicategory $\Pi_2(T)$ encodes the 2-homotopy type of T and it comes with the very nice feature that every arrow and 2-arrow is invertible. This special kind of bicategory is called a 2-groupoid.

2.5 The embedding Mon $\to$ Bicat

What we are going to do here, is to consider a canonical way of building a bicategory from a monoidal category. This process is a kind of vertical categorification of the notion of monoidal category, exactly as we vertically categorify groups : Grp $\to$ Cat.

Let $(M, \otimes)$ be a monoidal category. The associated bicategory $\mathcal{B}$ will described as a bicategory with one object enriched in M :

- $\mathrm{Ob}(\mathcal{B}) = \{*\}$;
- $\mathrm{Hom}(*,*) = \mathrm{Ob}(M)$;
- $2\mathrm{Hom}_{\mathcal{B}}(x,y) = \mathrm{Hom}_M(x,y)$ for objects x and y in M ;
- for x and y elements of $\mathrm{Hom}(*,*)$, composition if given by $y \circ x$ to be $x \otimes y$.

Moreover, as the monoidal structure on M satisfies the Pentagon and Triangle Identities, this is the same for $\mathcal{B}$.

2.6 2Cob^{ext}

A Topological Field Theory is a special functor from the category nCob to the category Vect – or more generally to any [insert your favorite structure adjectives] [insert your favorite categorical number]-category. The latter is very well understood but the former remains mysterious. The better we are be able to describe about the structure of nCob, the easier it will be to describe TFTs.

Actually we would like to get a description of the cobordism category with generators and relations. This is not possible if we take closed manifolds as objects. Instead, the idea would be to have manifolds with boundaries as generators. That means *extrending* nCob [7].

For the sake of simplicity, let's take $n = 2$ and try to describe what we would like 2Cob^{ext} to be as a bicategory :

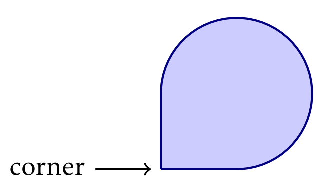
- Ob(2Cob^{ext}) is the class of all zero dimentional oriented compact manifolds ;
- $\mathrm{Hom}(X,Y)$ is the class of all oriented compact one dimensional cobordisms between X and Y ;
- $2\mathrm{Hom}(M,N)$ is the class of all oriented compact two dimensional cobordisms between cobordisms M and N.

Unfortunatly, building a bicategorical structure on 2Cob does not come for free ; we have to face several problems.

Problem 1 : $\partial \circ \partial = 0$

The first problem comes from the well-know property of manifolds with boundaries : *the boundary of the boundary is empty*. Hence, in most cases there are no 2-arrows !

The solution is to use *manifolds with corners*. Here is an example :



The basic idea is that, while a 2-manifold with boundary locally looks like an open subset of $\mathbb{R}\times\mathbb{R}_+$, a 2-manifold with corners locally looks like an open subset of $\mathbb{R}_+\times\mathbb{R}_+$. The two topological spaces are not homeomorphic as you can see by looking at their boundaries : $\partial(\mathbb{R}_+\times\mathbb{R}_+)\simeq\mathbb{R}_+\times\{0,1\}\Rightarrow\partial\partial(\mathbb{R}_+\times\mathbb{R}_+)\neq\emptyset$ whereas $\partial(\mathbb{R}\times\mathbb{R}_+)=\mathbb{R}\Rightarrow\partial\partial(\mathbb{R}\times\mathbb{R}_+)=\emptyset$. Apart from that, all the definitions are the same as for manifolds with boundaries.

So, we modify the definition to :

- 2Hom(M,N) would be the class of all compact cobordisms with corners between M and N.

Problem 2 : compositions in the smooth setting

We would like the composition of two cobordisms to be obtained by gluing. In the topological setting, this is no problem : the resulting space has a canonical topological structure for free.

This is no longer true in the smooth setting. To define a smooth structure on the gluing of two cobordisms, one needs to specify a *tubular neighborhood* or *collar*. Let C and C' be two manifolds with a common boundary B, then a differential structure on $C\cup_B C'$ is equivalent to the data of an embedding $B\times\mathbb{R}\to C\cup_B C'$ that restricts to smooth embeddings $B\times[0,+\infty)\to C$ and $B\times(-\infty,0]\to C'$. Nevertheless, the resulting smooth manifolds are diffeomorphic, but by a non canonical diffeomorphism.

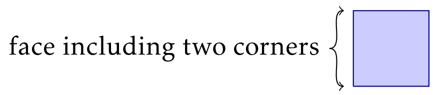
The fact that this diffeomorphism is non-canonical is not a problem when defining the composition of 1-arrows, as we allow them to be associative up to a 2-arrow. However for 2-arrows, composition has to hold in a strict sense and we are forced to change again our definition :

- 2Hom(M,N) would know be the class of all diffeomorphic class of two dimensional compact cobordisms with corners between M and N.

Moreover, not every manifold with corner admits a tubular neighborhood. The manifold M in the picture has none, because $\partial M\times\mathbb{R}_+$ has a sharp edge and thus, cannot be diffeomorphic, as manifold with corners, with an open subset of M.

Somehow, the class of manifolds with corner is too large for our interest. In fact, we are mainly focused on the study of manifolds that we slice and glue. A manifold sliced twice becomes a manifold with corner. But here every corner would be the result of at least two slicings ; we are lead to the good notion of *manifold with faces*.

The disk with a corner in the picture above is the typical manifold with corner that is not a manifold with faces. Morally, any corner should be at the intersection multiple faces whereas in the counter-example, the corner belongs to one face only. Here is an example of manifold with faces :

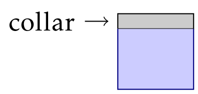


Definition 2.2 *Let M be a manifold with corners, for $p \in M$, the index of p is the number of vanishing coordinates of $\varphi(p)$ for any chart φ.*

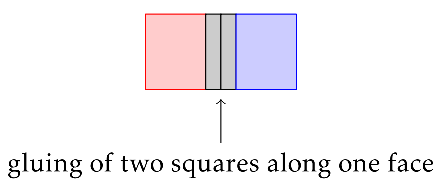
A connected face of a manifold with corners M is the closure of a component of $\{p \in M / index(p) = 1\}$. A manifold with corners is a manifold with faces if each $p \in X$ belongs to index(p) faces. For such a manifold a face *is a disjoint union of connected faces, and is itself a manifold with faces.*

Hence, a face of a manifold is the closure of a usual component of the boundary. This is the good notion to replace the notion of boundary which fails in the setting of manifolds with corners. Indeed, the boundary of the disk with a corner is not a manifold with corners... Moreover, the notion of *face* will help us to broaden the notion of notion of *collar* : we'll be able to glue faces to faces which is less restrictive than gluing boundaries to boundaries.

An example of a collar of a face :



and an example of gluing :



All this said, we obtain good theorems for collars and gluings [2]. So we are now able to give the right definition of objects and arrows of our category.

Definition 2.3 *Let Y_0 and Y_1 be 0-manifolds. A* 1-bordism *from Y_0 to Y_1 is a smooth compact 1-manifold with boundary, W, equipped with a decomposition and isomorphism of its boundary $\partial W = \partial_{in} W \sqcup \partial_{out} W \cong Y_0 \sqcup Y_1$.*

Definition 2.4 *Let Y_0 and Y_1 be 0-manifolds and let W_0 and W_1 be two 1-bordisms from Y_0 to Y_1. A* 2-bordism *consists of a compact 2-dimensional manifold with faces, S, with a decomposition of ∂S into two faces $\partial_0 S$, $\partial_1 S$ and equipped with the following additional structures:*

- *A decomposition and isomorphism:*

$$\partial_0 S = \partial_{0,in} S \sqcup \partial_{0,out} S \xrightarrow{g} W_0 \sqcup W_1.$$

- *A decomposition and isomorphism:*

$$\partial_1 S = \partial_{1,in} S \sqcup \partial_{1,out} S \xrightarrow{f} Y_0 \times I \sqcup Y_1 \times I.$$

These are required to induce isomorphisms

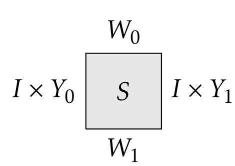
$$\begin{aligned} f^{-1}g :& \partial_{in} W_0 \sqcup \partial_{out} W_0 \to Y_0 \times \{0\} \sqcup Y_1 \times \{0\} \\ f^{-1}g :& \partial_{in} W_1 \sqcup \partial_{out} W_1 \to Y_0 \times \{1\} \sqcup Y_1 \times \{1\}, \end{aligned}$$

which coincide with the structure isomorphisms of W_0 and W_1.

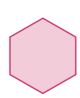
Two 2-bordisms, S and S', are isomorphic if there is a diffeomorphism $h : S \to S'$, which restricts to diffeomorphisms,

$$\begin{aligned} \partial_{0,in} S &\to \partial_{0,in} S' \\ \partial_{0,out} S &\to \partial_{0,out} S' \\ \partial_{1,in} S &\to \partial_{1,in} S' \\ \partial_{1,out} S &\to \partial_{1,out} S' \end{aligned}$$

and such that $f' \circ h = f$ and $g' \circ h = g$.



Remark *In this definition, the decomposition of $\partial_1 S$ ensures that a 2-cobordism as the minimal possible number of corners. For example :*

 $\notin$ 2Hom([0, 1], [0, 1])

This is compulsory to make sure that horizontal composition of 2-arrows is possible.

Two more ingredients are needed, but we shall not discuss them here. One needs the notion of *pseudo-isotopy* to kick off corners after gluing and one needs to prove that any diffeomorphism $X \to Y$ can be promoted to a cobordism between X and Y. See [2].

Theorem 2.5 $2\textsc{Cob}^{ext}$ *is a bicategory.*

Remark *If one consider the disjoint union of manifolds as an operation, then* $2\textsc{Cob}^{ext}$ *becomes a symetric monoidal bicategory. With a bit of care, it is possible to describe* $n\textsc{Cob}$ *as a symmetric monoidal bicategory in the same way.*

Besides, this construction has two drawbacks : first, to define the horizontal composition of two 1-morphisms, we need the axiom of choice to get a collar with each manifold. More seriously, we are not able to add more structure on the manifolds, especially orientation. The solution to this problem lies in the notion of halos *[2].*

3 Two main properties of bicategories

3.1 The Coherence Theorem

It is now time to justify the requirement of the Pentagon and Triangle Identities in the definition of bicategories. In the associative setting, there is a theorem claiming that you can write $a \circ b \circ c$ for $(a \circ b) \circ c$ because all the bracketings are equal. The Coherence Theorem is an analogue in the setting of associativity up to an invertible 2-arrow.

A *bracketing* consists of a parenthesized word in "$-$" (dashes) and "1" (ones). The *length* of a bracketing is the number of dashes. More precisely, we make the following recursive definition: Both (1) and $(-)$ are bracketings (of length zero and one, respectively). In u and v are bracketings, then $(u)(v)$ is a bracketing of length length(u)+length(v). The examples below have lengths five and four respectively.

$$((--)1)(((--)1)(1-)) \qquad \text{and} \qquad ((((11)1)-)((--)1))-$$

Let $\mathcal{A}$ be a bicategory. Given a bracketing b of length n, and a word w of composable 1-morphism from $\mathcal{A}$ of length n, then composition in $\mathcal{A}$ gives a canonical 1-morphism in $\mathcal{A}$, which we denote $b(w)$. Define the following elementary moves on bracketings:

1. (Deleting Identities) $1b \Leftrightarrow b$ and $b1 \Leftrightarrow b$
2. (Rebracketing) $(bb')b'' \Leftrightarrow b(b'b'')$

where b, b', b'' represent arbitrary bracketings. An elementary move has an apparent source and target which are bracketings. A *path* also has source and target bracketings and is defined recursively as follows:

1. Elementary moves are paths,
2. If p and p' are paths such that the source of p is the target of p', then $p \circ p'$ is a path with source the source of p' and target the target of p.
3. if p and p' are paths, then $(p)(p')$ is a path with source (source(p))(source(p')) and target (target(p))(target(p')).

Given two bracketings b and b' of length n and a word w of composable 1-morphism from $\mathcal{A}$, also of length n, we have the two 1-morphisms $b(w)$ and $b'(w)$ of $\mathcal{A}$. Given a path s starting at b and ending at b', we get a canonical 2-morphism of $\mathcal{A}$, $s(w) : b(w) \to b(w')$, given by replacing the elementary moves by the appropriate unitors and associators from $\mathcal{A}$.

Theorem 3.1 (MacLane's Coherence Theorem [3]) *If b and b' are two bracketings of length n and w is a word of composable 1-morphism from the bicategory $\mathcal{A}$, also of length n, then there exists a path $s : b \to b'$ starting at b and ending at b'. Moreover, given any two such paths $s, s' : b \to b'$, the resulting 2-morphisms $s(w) : b(w) \to b(w')$ and $s'(w) : b(w) \to b(w')$ are identical.*

3.2 Strictification

A crucial fact about bicategories is that *any* small bicategory is equivalent to a 2-category in a canonical way. This is no longer true for tricategories (see the introduction of [4]). But before talking of equivalence between bicategories, we should first define what a morphism or functor between bicategories is.

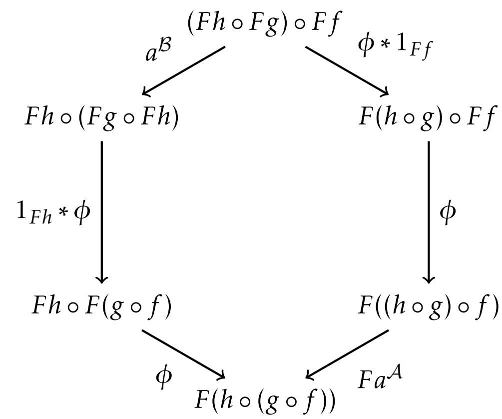
Definition 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be bicateogries. A* homomorphism $F : \mathcal{A} \to$ *$\mathcal{B}$ consists of the data:*

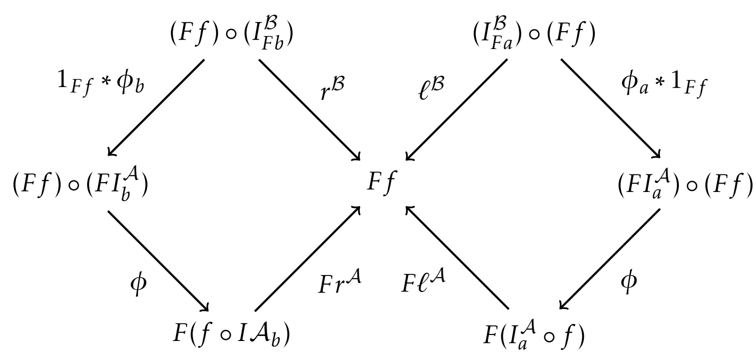
1. *A function $F : Ob(\mathcal{A}) \to Ob(\mathcal{B})$,*
2. *Functors $F_{ab} : \mathcal{A}(a, b) \to \mathcal{B}(F(a), F(b))$,*
3. *Natural isomorphisms*

$$\phi_{abc} : c^{\mathcal{B}}_{F(a)F(b)F(c)} \circ (F_{bc} \times F_{ab}) \to F_{ac} \circ c^{\mathcal{A}}_{abc}$$
$$\phi_a : I^{\mathcal{B}}_{F(a)} \to F_{aa} \circ I^{\mathcal{A}}_a$$

(thus invertible 2-morphisms $\phi_{gf} : Fg \circ Ff \to F(g \circ f)$ and $\phi_a : I^{\mathcal{B}}_{Fa} \to F(I^{\mathcal{A}}_a)$).

such that the following diagrams commute:





If the natural isomorphisms ϕ_{abc} and ϕ_a are identities, then the homomorphism F is called a strict homomorphism.

These definitions allow us to define the category $2\mathbf{cat}_s$ of all small 2-categories with strict morphisms and the category **bicat** of all small bicategories with weak morphisms.

Achtung ! These two categories are *not* equivalent.

Back to basics, if $\mathcal{C}$ is a 2-category then it is also a bicategory. This gives us an inclusion functor $\iota : 2\mathbf{cat}_s \to \mathbf{bicat}$. What we are going to do is to construct a left adjoint to ι. MacLane's coherence theorem allows us to form the following construction. Define the *standard bracketing* b^n_{std} of length n to be the bracketing:

$$((\cdots(((--)-)-)\cdots)-).$$

Given a bicategory $\mathcal{A}$ we define the following associated 2-category, $Q(A)$:

- The objects of $Q(\mathcal{A})$ are the same as those of $\mathcal{A}$.
- The 1-morphisms of $Q(\mathcal{A})$ are the words of composable 1-morphisms in $\mathcal{A}$.
- Given two such words w, w', the 2-morphisms are the 2-morphisms between $b_{\mathrm{std}}^{|w|}(w)$ and $b_{\mathrm{std}}^{|w'|}(w')$.

Vertical composition coincides with that in $\mathcal{A}$. The horizontal composition of 1-morphisms is given by concatenating words, and hence is strictly associative. The horizontal composition of 2-morphisms is given by using the canonical rebracketing 2-morphisms supplied by MacLane's coherence theorem. Given two 2-morphisms $\alpha : b_{\mathrm{std}}(w) \to b'_{\mathrm{std}}(w')$ and $\beta : b_{\mathrm{std}}^{|\tilde{w}|}(\tilde{w}) \to b_{\mathrm{std}}^{|\tilde{w}'|}(\tilde{w}')$, their horiszontal composition is defined to be the composite:

$$b_{\mathrm{std}}^{|w\tilde{w}|}(w\tilde{w}) \to b_{\mathrm{std}}^{|w|}(w) \circ b_{\mathrm{std}}^{|\tilde{w}|}(\tilde{w}) \stackrel{\alpha*\beta}{\to} b_{\mathrm{std}}^{|w'|}(w') \circ b_{\mathrm{std}}^{|\tilde{w}'|}(\tilde{w}') \to b_{\mathrm{std}}^{|w'\tilde{w}'|}(w'\tilde{w}')$$

where the unlabeled arrows are the canonical 2-morphisms from MacLane's theorem.

This composition is automatically associative and hence defines a 2-category $Q(\mathcal{A})$, called the *strictification* of $\mathcal{A}$. In fact Q defines a functor,

$$Q : \mathbf{bicat} \to 2\mathbf{cat}_s$$

The unit and co-unit of the adjunction (i.e. $Q \circ \iota$ and $\iota \circ Q$) give us for each 2-category $\mathcal{B}$ a strict homomorphism $Q(\mathcal{B}) \to \mathcal{B}$ (which we can take to be the homomorphism which sends a composable word of 1-morphisms in $\mathcal{B}$ to its composition) and for each bicategory $\mathcal{A}$ a weak homomorphism $\mathcal{A} \to Q(\mathcal{A})$ (which we can take to be the homomorphism which sends a 1-morphism in $\mathcal{A}$ to the singleton word consisting of exactly that 1-morphism). Both of these homomorphisms are equivalences of bicategories[2]. Note, however, that for a 2-category $\mathcal{B}$, the inverse equivalence to the canonical strict homomorphism $Q(\mathcal{B}) \to \mathcal{B}$ is nearly always a *weak* homomorphism.

Theorem 3.3 *Every small bicategory is equivalent to a 2-category.*

References

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[2] Christopher Schommer-Pries, *The Classification of Two-Dimensional Topological Field Theories*, preprint arXiv:1112.1000v1.

[2] In the sense given by the structure of tricategory of small bicategories Bicat, see [2].

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