Claudio Sibilia

Symmetric monoidal bicategory

Recall the definition of symmetric monoidal bicategory $\mathcal{C}$. It is a category with a functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ and 4 natural isomorphisms

$\alpha : (a\otimes b)\otimes c \longrightarrow a\otimes(b\otimes c)$ "associativity constraints"

$l : 1\otimes a \longrightarrow a$ "left unit " [1 is an object in $\mathcal{C}$]

$r : a \longrightarrow a\otimes 1$ "right unit "

$\beta : a\otimes b \longrightarrow b\otimes a$ "braiding"

which satisfy the following relations

P_1 : (Pentagon axiom)

$$\begin{tikzcd}
& (a\otimes b)\otimes(c\otimes d) \arrow[dr, "\alpha"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "\alpha"] \arrow[d, "\alpha\otimes I"'] & \circlearrowleft & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[rr] & & a\otimes((b\otimes c)\otimes d) \arrow[u, "I\otimes\alpha"']
\end{tikzcd}$$

T_1:

$$\begin{tikzcd}
(a\otimes 1)\otimes b \arrow[r, "\alpha"] & a\otimes(1\otimes b) \arrow[d, "I\otimes l"] \\
a\otimes b \arrow[u, "r\otimes 1"] \arrow[r, bend right, "I_{a\otimes b}"'] \arrow[ur, phantom, "\circlearrowleft"] & b\otimes a
\end{tikzcd}$$

T_2:

$$\begin{tikzcd}
(1\otimes a)\otimes b \arrow[r, "l\otimes 1"] \arrow[d, bend right, "\alpha"'] & a\otimes b \\
1\otimes(a\otimes b) \arrow[ur, bend right, "l"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

T_3:

$$\begin{tikzcd}
(a\otimes b) \arrow[r, "I\otimes r"] \arrow[d, bend right, "r"'] & a\otimes(b\otimes 1) \\
(a\otimes b)\otimes 1 \arrow[ur, bend right, "\alpha"'] \arrow[u, phantom, "\circlearrowleft"] &
\end{tikzcd}$$

B_1:

$$\begin{tikzcd}
a\otimes(b\otimes c) \arrow[rr, "\beta"] & & ((b\otimes c)\otimes a) \arrow[d, "\alpha"] \\
(a\otimes b)\otimes c \arrow[u, "\alpha"] \arrow[d, "\beta\otimes I"'] & \cong & b\otimes(c\otimes a) \\
(b\otimes a)\otimes c \arrow[rr, "\alpha"] & & b\otimes(a\otimes c) \arrow[u, "I\otimes\beta"']
\end{tikzcd}$$

B_2:

$$\begin{tikzcd}
(a\otimes b)\otimes c \arrow[rr, "\beta"] & & c\otimes(a\otimes b) \arrow[d, "\alpha^{-1}"] \\
a\otimes(b\otimes c) \arrow[u, "\alpha^{-1}"] \arrow[d, "I\otimes\beta"] & \cong & (c\otimes a)\otimes b \\
a\otimes(c\otimes b) \arrow[rr, "\alpha^{-1}"'] & & (a\otimes c)\otimes b \arrow[u, "\beta\otimes I"']
\end{tikzcd}$$

B_3:

$$\begin{tikzcd}
a\otimes b \arrow[rr, "I_{a\otimes b}"] \arrow[dr, "\beta"'] & & a\otimes b \\
& b\otimes a \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Concretely this axioms say: "sequence of bracketing, flip, $\otimes 1$ play no role in the end".

What we want to do now is try to "extend" definition for Bicategories. The strategy is the following:

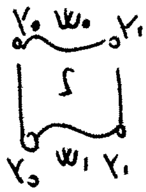
- look at some properties of $2\,Cob(d)^{ext}$,
  - look at $Cob(d-1)$ to define on $2\,Cob(d)^{ext}$ a "coarse" symmetric monoidal structure,
  - extracht by $2\,Cob(d)^{ext}$ a general definition which holds for every bicategory.

1. Some properties of $2\,Cob(d)^{ext}$

Recall that $2\,Cob(d)^{ext}$ is a bi category where

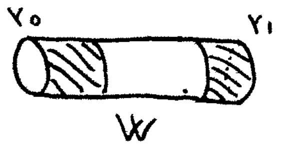
- the objects are closed $(d-2)$ manifolds.
- the morphism category $Cob_{d-1}(Y_0, Y_1)$ is the class of smooth $d-1$-manifolds with boundary s.t. $\partial W = \partial_{in} W \amalg \partial_{out} W \simeq Y_0 \amalg Y_1$ $(*)$
- the morphism in $Cob_{d-1}(Y_0, Y_1)$ are the equivalence classes of smooth (d)-manifolds with faces equipped with two isomorphisms of $d-1$ manifold s.t. if S is a morphism between $W_0, W_1 \in Cob_{d-1}(Y_0, Y_1)$ then

the "lower" and the "upper" boundary of S are isomorphic to W_0, W_1 and the "right" and the left boundary of S are isomorphic to $Y_0 \times I$, $Y_1 \times I$. Moreover this isomorphism extend (*).



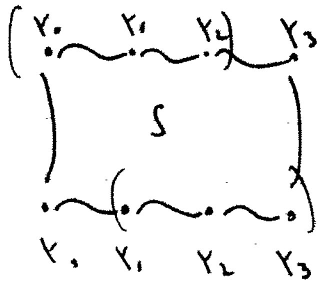
Rmk ①: the fact that the objects in $Cob_{d-1}(Y_0,Y_1)$ are "1-bordism" between Y_0, Y_1 and not "equivalence class of 1-bordism" generates some problems. By Pentagon identity in Damien Legoy Talks (last week) we want that the composition of 1-morphisms is associative up a canonical isom (denoted by a). To do that in $Cob_{d-1}(Y_0,Y_1)$ we need to do the following things:

① Equip every 1-bordism with a collar around the boundary:



(Fixed!)

② Note that the composition of 1-bordism is associative but only up to a diffeomorphism (= 2-morphism)



Rmk ② Let $f : X \to Y$ be a map between two d-manifolds. Then f can be associated to a 1 bordism as follows:

$W_f := X \times I$

the isom on the boundary is $\partial W_f = X \times \{0\} \amalg X \times \{1\} \xrightarrow{id \amalg f} X \amalg Y$

this idea can be extended for 2-bordism.

2. "Coarse" Symmetric monoidal bicategory

Since we have the following conceptual inclusion $Cob(d+1) \text{ "}\subset\text{" } 2Cob(d)^{ext}$

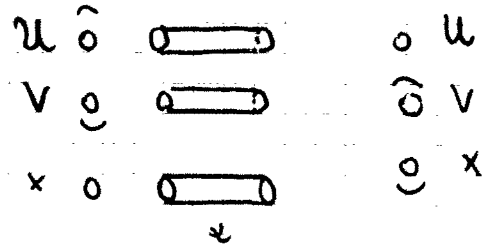
Our standard candidates for $\otimes$ and 1 are $\amalg$ and $\emptyset$.

Thus we define $\amalg$ and $\emptyset$ for $\otimes$ and $\amalg$ in $2Cob(d)^{ext}$. They are well-defined exactly as for $Cob(d-1)$. Now we define the map a but only at the level of objects and 1-morphism (Note that this "much" equivalent to $Cob(d+1)$)

thus a is the map

$$a : \left(2Cob(d)^{ext} \otimes 2Cob(d)^{ext}\right) \otimes 2Cob(d)^{ext} \to 2Cob(d)^{ext} \otimes \left(2Cob(d)^{ext} \otimes 2Cob(d)^{ext}\right)$$

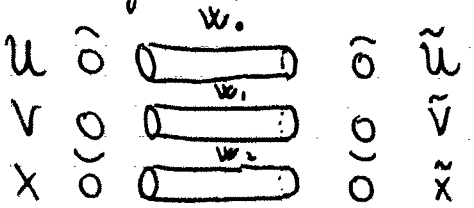
① at the level of objects:



Sending:

$$\begin{array}{ccccc} (U \times (1,1)) & \times & (V \times (1,2)) \times & & (X \times (2,1)) \\ \downarrow & & \downarrow & & \downarrow \\ (U \times (2,1)) & \times & (V \times (1,1)) & & (X \times (2,2)) \end{array}$$

② at the level of 1-morphism



Sending:

$$\begin{array}{ccccc} (W_0 \times (1,1)) & \times & (W_1 \times (1,2)) & \times & W_2 \times (2,1) \\ \downarrow & & \downarrow & & \downarrow \\ (W_0 \times (2,1)) & \times & (W_1 \times (1,1)) & \times & (W_2 \times (1,2)) \end{array}$$

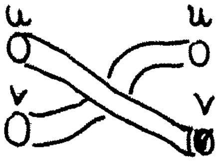
<u>C.P:</u> a agrees with the associator map in $Cob(d+1)$. Now I suppose that a should satisfy a Pentagon relation P_1. If I write this equation at the level of the objects I have:

$$\begin{tikzcd}
 & (a\otimes b)\otimes(c\otimes d) \arrow[dr, "a"] & \\
((a\otimes b)\otimes c)\otimes d \arrow[ur, "a"] \arrow[d, "a\otimes I"'] \arrow[rr, phantom, "(\text{up to } \pi \Uparrow)"] & & a\otimes(b\otimes(c\otimes d)) \\
(a\otimes(b\otimes c))\otimes d \arrow[rr, "a"'] & & a\otimes((b\otimes c)\otimes d) \arrow[u, "I\otimes a"']
\end{tikzcd}$$

<u>But:</u> by rmk ② i can see α as a 1-morphism. thus by rmk ① I have that the above diagramm is commutative only up to a invertible 2-morphism. I write this 2-morphisms π_{abcd}. It is a family of two morphism which depend by a,b,c,d.

Following the above example we define l, r, β only at the level of the objects and 1-morphism such that l, r, β agrees with "classical" l, r, β in $Cob(d+1)$

Ex: β is



thus as above we can view l, r, β as 1-morphism. Thus if I compute $T_1, T_2, T_3, B_1, B_2, B_3$ using the 1-morphism l, r, β I obtain that such a diagrammes are commutative up to a family of 2-morphisms. More precisely

- From: | I obtain | direction arrow

From:	I obtain	direction arrow
P_4	π_{abcd}	$\uparrow$
T_1	μ_{ab}	$\downarrow$
T_2	λ_{ab}	$\downarrow$
T_3	ρ_{ab}	$\downarrow$
B_1	R_{abc}	$\downarrow$
B_2	S_{abc}	$\downarrow$
B_3	σ_{ab}	$\downarrow$

Thus in the end we have that the structure of the S.M.B. is given by

1) $\otimes$ = " Map between category" +

2) a, l, r, β = " Family of maps between object" +

3) $\pi, \mu, h, \ldots$ = " Fafamily of maps which acts on the maps ②" +

$\Big\} + P_4, \ldots, B_3$

Thus we want: 3 level of maps between bicategory wich satisfies $P_4, \ldots, B_3$
in particular a, l, r, β are "natural trasformations" in a bicategory between the map $\otimes$ and π, μ, h are map which act on the natural trasformation.

③ Map between bicategory

In this part (finally) I give you the definition of map like $\otimes$, e, π.
First we start with the "functor" in bicategory:

Def: Let A, B be bicategories. A homomorphism $F: A \to B$ consist of the data:

(1) A function $F: \text{ob}A \to \text{ob}B$

(2) Functors $F_{ab}: A(a,b) \to B(F(a),F(b))$

(3) Natural isom

$$\phi_{abc}: \quad C^B_{F(a)F(b)F(c)} \circ (F_{bc} \times F_{ab}) \longrightarrow F_{ac} \circ C^A_{abc}$$

$$\phi_a: \quad I^B_{F(a)} \longrightarrow F_{aa} \circ I^A_a$$

(Thus invertible 2-morphism $\phi_{gf}: Fg \circ Ff \to F(g \circ f)$

$$\phi_a: \quad I^B_{Fa} \longrightarrow F(I^A_a) \,)$$

s.t the diagramm in the Talks of Damien (last week) commutes.

Ex: · $\otimes$ in $2Cob^{ext}(\mathcal{C})$, $\phi_{gf}: F(g_1,g_2) \circ F(f_1,f_2) \longrightarrow F((g_1,g_2)\circ(f_1,f_2))$ means: $(g_1 \otimes f_1)\circ(f_2 \otimes f_2) \cong g_1 \circ f_1 \otimes g_2 \circ f_2$

Notation: Let A be a bicategory and let $\sigma: a \to b$ be a 1-morphism in A.
Pre an Post compositions define functors:

$$\sigma^*: A(b,c) \longrightarrow A(a,c)$$

$$\sigma_*: A(d,a) \longrightarrow A(d,b)$$

Now we want to see a special type of maps which are very related to natural trasformation between two functors in category but in this case they don't define necessarly an equivalence to bicategory.

Def: Let (F,ϕ), $(G,\psi): A \to B$ be hom between bicategory.
A trasformation $\sigma: F \to G$ is given by

(1) 1-morphism $\sigma_a: Fa \to Ga \quad \forall a \in A$

(2) Natural isom $\sigma_{ab}: (\sigma_a)^* \circ G_{ab} \longrightarrow (\sigma_b)_* \circ F_{ab}$

(inv. 2-morphism $\sigma_f: Gf \circ \sigma_a \longrightarrow \sigma_b \circ Ff \quad \forall f \in A$)

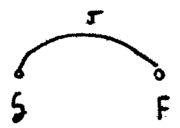
such that the following diagramm commute $\forall\, f: a \to b,\ g: b \to c$:

$$
\begin{tikzcd}
 & \mathcal{G}(g\circ f)\circ\sigma_a \arrow[dr, "\sigma_{gf}"] & \\
(\mathcal{G}g\circ\mathcal{G}f)\circ\sigma_a \arrow[ur, "\psi_{g,f}*id_{\sigma_a}"] \arrow[d, "a^B"'] & & \sigma_c\circ F(g\circ f) \arrow[d, "id_{\sigma_c}*\phi_{g,f}"] \\
\mathcal{G}g\circ(\mathcal{G}f\circ\sigma_a) \arrow[d, "id_{\mathcal{G}b}*\sigma_f"'] & \circlearrowright & \sigma_c\circ(Fg\circ Ff) \arrow[d, "a^B"] \\
\mathcal{G}g\circ(\sigma_b\circ Ff) \arrow[dr, "(a^B)^{-1}"'] & & (\sigma_c\circ Fg)\circ Ff \arrow[dl, "\sigma_g*id_{Ff}"] \\
 & (\mathcal{G}g\circ\sigma_b)\circ Ff &
\end{tikzcd}
$$

<u>Ex</u>: Consider the map a between $\otimes(\otimes\times id) = F$ and $\otimes(id\times\otimes) = \mathcal{G}$ sending $\left(2Cob(d)^{ext}\right)^4$ to $\left(2Cob(d)\right)^{ext}$. The 1 morphism $\sigma_x: F(x) \to \mathcal{G}(x)$, $x \in \left(2Cob(d)\right)^{ext}$ is the 1-bordism given by x. Since a can be defined as a map at the level of our bordisms, then it induces a <u>tras-morphism</u> which is the ②.

It is related to natural trasformation because the picture is

$$
\begin{tikzcd}
A \arrow[r, "F"] \arrow[d, phantom, "\Downarrow\sigma"] & B \\
A \arrow[r, "\mathcal{G}"'] & B
\end{tikzcd}
\quad \approx
$$



Now we define the deformation of trasformation exactly as "a is deformed by π"

<u>Def</u>: Let (F,ϕ); $(\mathcal{G},\psi): A \to B$ be two functors between bicategory, $\sigma, \theta: F \to \mathcal{G}$ be two trasformation. A modification $\Gamma: \sigma \to \Theta$ consist of 2-morphisms $\Gamma_x: \sigma_x \to \theta_x$ for every object $x \in A$ such that

$$
\begin{tikzcd}
\mathcal{G}f\circ\sigma_a \arrow[r, "id*\Gamma_a"] \arrow[d, "\sigma_f"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{G}f\circ\theta_a \arrow[d, "\theta_f"] \\
\sigma_b\circ Ff \arrow[r, "\Gamma_b*id"'] & \theta_b\circ Ff
\end{tikzcd}
$$

$\forall$ 1-morph $f: a \to b$

<u>exemple</u>: • $F = \otimes(\otimes \times id)$, $G = \otimes(id \times \otimes)$ hom.

• trasformations a^2, $(I\otimes a(a(a\otimes I)))$

then π_{abcd} is a <u>modifications</u>

<u>Picture:</u>

$$\begin{tikzcd} F \arrow[r, bend left=50, "\sigma"] \arrow[r, bend right=50, "\theta"'] \arrow[r, phantom, "\Downarrow\Gamma"] & G \end{tikzcd}$$

we can actually show that $\mathrm{Bicat}(A,B)$ is a bicategory where:

• ob = hom $A \to B$

• 1-morph = trasformation

• 2-morph = modification.

<u>Now:</u> We can state the definition of symmetric monoidal bicategory

<u>Def</u>: A S.M.B consist of a bicategory M together with the following data:

a) $1 \in M$ (a distinguished object)

b) a homomorphism

$$\otimes = (\otimes, \phi^{\otimes}_{(f,f'),(g,g')}, \phi^{\otimes}_{(a,a')}) : M \times M \to M$$

c) trasformations

$$\begin{cases} a = (a_{abc}, a_{fgh}) : (a\otimes b)\otimes c \to a\otimes(b\otimes c) \\ l = (l_a, l_f) : 1\otimes a \to a \\ r = (r_a, r_f) : a \to a\otimes 1 \end{cases}$$

$$\{ \beta = (\beta_{ab}, \beta_{fg}) : a\otimes b \to b\otimes a$$

For wich there exist an inverse.

d) invertible modifications:

· π, μ, λ, ρ ar in P_1, T_1, T_2, T_3

· R, S ar in $B_1 B_2$

· σ ar in B_3

s.t they satisfy: • (TA 1), (TA2), (TA3) for $(M, 1, \otimes, a, l, r, \pi, \mu, \lambda, \rho)$

• (BA1), (BA2), (BA3), (BA4) " (β, R, S)

• (SA1), (SA2) " " and σ

• (SMA)

Examples:

1) CAT:

- $Ob(Cat)$ = class of all small categories
- $Hom(X,Y) = Fun(X,Y)$
- $2Hom(X,Y) = Nat(F,G)$ " natural transformation"

Let $\otimes = \times$, then define α, β, l, r similar as in $2Cob^{ext}(\cdot\,)$.
in this case the modifications are trivial

2) Set

- $Ob(Set)$ = class of sets
- $Hom(X,Y)$ = set of relations between X and Y
- $2Hom(P,\Theta)$ = set of one element if $P \subset \Theta$

$\otimes = \times$, the approach is the same as above.

3) Alg with bimodules

- $Ob(Alg)$ = class of all algebras
- $Hom(A,B)$: is the class of all A-B-modules, that is the class of all abelian groups M s.t. $A \circlearrowright M \circlearrowleft B$
- $2Hom(M,N)$ = bimodule hom

Let: $\otimes = \otimes_K$ (tensor product of algebra)
we define the action of $A \otimes B$ as: $a \cdot b(m) = a \circ m \circ b$
Thus:

$$\big(\underset{M}{Hom(A,B)},\ \underset{N}{Hom(C,D)}\big) \longrightarrow M \otimes N$$

where

$$A \otimes B \circlearrowright M \otimes N \circlearrowleft C \otimes D$$

in this case modification and transformation are NOT TRIVIAL.

To be symm monoidal. →

$$\text{(SMA)}\quad \begin{tikzcd} x\otimes y \arrow[r, "\beta"] \arrow[d, "\beta"'] \arrow[dr, "I"] & y\otimes x \\ y\otimes x \arrow[r, "\beta"'] \arrow[ur, phantom, "\Downarrow\sigma" near start] & x\otimes y \arrow[u, "\beta"'] \end{tikzcd} \quad = \quad \begin{tikzcd} x\otimes y \arrow[r, "\beta"] \arrow[d, "\beta"'] & y\otimes x \\ y\otimes x \arrow[r, "\beta"'] \arrow[ur, "I"] \arrow[ur, phantom, "\Downarrow\sigma" near end] & x\otimes y \arrow[u, "\beta"'] \end{tikzcd}$$

FIGURE 4. Equation (SMA)

$$\begin{tikzcd} & Ha\otimes(Hb\otimes Hc) \arrow[r, "I\otimes\chi"] & Ha\otimes H(b\otimes c) \arrow[dr, "\chi"] & \\ (Ha\otimes Hb)\otimes Hc \arrow[ur, "\alpha'"] \arrow[dr, "\chi\otimes I"'] & & \Downarrow\omega & H(a\otimes(b\otimes c)) \\ & H(a\otimes b)\otimes Hc \arrow[r, "\chi"'] & H((a\otimes b)\otimes c) \arrow[ur, "H\alpha"'] & \end{tikzcd}$$

$$\begin{tikzcd} & H(1)\otimes Ha \arrow[r, "\chi"] \arrow[d, phantom, "\Downarrow\gamma"] & H(1\otimes a) \arrow[dr, "H\ell"] & \\ 1'\otimes Ha \arrow[ur, "\iota\otimes I"] \arrow[rrr, "\ell'"'] & & & Ha \end{tikzcd} \qquad \begin{tikzcd} & Ha\otimes 1' \arrow[r, "I\otimes\iota"] \arrow[d, phantom, "\Uparrow\delta"] & Ha\otimes H(1) \arrow[dr, "\chi"] & \\ Ha \arrow[ur, "r'"] \arrow[rrr, "Hr"'] & & & H(a\otimes 1) \end{tikzcd}$$

FIGURE 5. Modifications for Monoidal Homomorphism (♣)

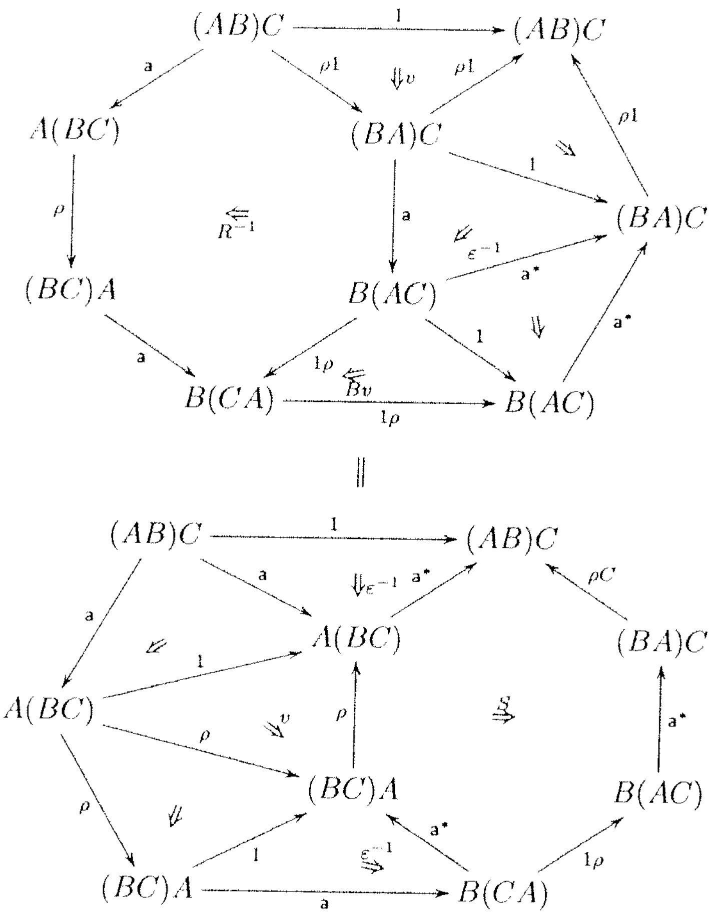
$$\begin{tikzcd} & H(b\otimes a) \arrow[dr, "H\beta"] & \\ H(b)\otimes H(a) \arrow[ur, "\chi"] \arrow[dr, "\beta'"'] \arrow[rr, phantom, "\Downarrow u"] & & H(a\otimes b) \\ & H(a)\otimes H(b) \arrow[ur, "\chi"'] & \end{tikzcd}$$

FIGURE 6. Modifications for Braided Monoidal Homomorphism (◊)

♣ The following equations hold, where these designations refer to the 2-morphisms in Figures 8, 9, 10, 11, 12, and 13.

$$(MBTA1.a) = (MBTA1.b),$$
$$(MBTA2.a) = (MBTA2.b),$$
$$(MBTA3.a) = (MBTA3.b).$$

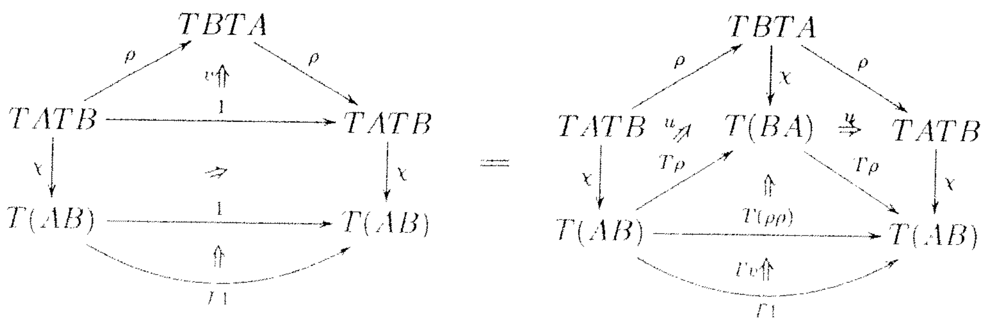
(SA2) for all objects A, B and C of $\mathcal{K}$ the following equation holds.



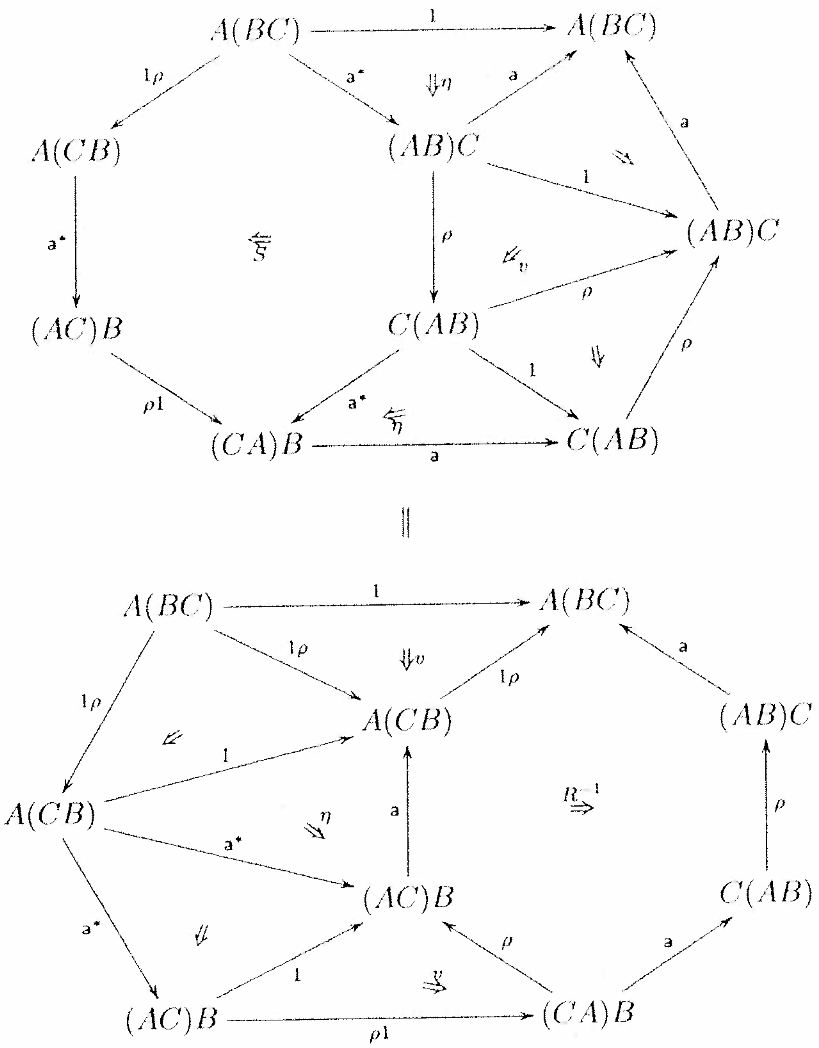
A *sylleptic monoidal bicategory* is a braided monoidal bicategory equipped with a syllepsis.

If $\mathcal{K}$ and $\mathcal{L}$ are sylleptic monoidal bicategories, then a *sylleptic weak monoidal homomorphism* $T\colon \mathcal{K} \to \mathcal{L}$ is a braided weak monoidal homomorphism $T\colon \mathcal{K} \to \mathcal{L}$ such that the axiom (SHA1) that follows holds.

(SHA1) For all objects A and B of $\mathcal{K}$ the following equation holds.



(SA1) for all objects A, B and C of $\mathcal{K}$ the following equation holds;



(BA4) for all objects A, B and C of $\mathcal{K}$ the following equation holds.

$$
\begin{tikzcd}
 & & B(CA) \arrow[r, "a^*"] \arrow[r, phantom, shift right=8ex, "\Downarrow s^{*-1}"] & B(CA) \arrow[dr, "a^*"] & & \\
 & B(AC) \arrow[ur, "1\rho"] \arrow[dr, "a^*"] \arrow[d, phantom, "\Downarrow\varepsilon"] & & & (CB)A \arrow[d, phantom, "\Rightarrow"] & \\
(BA)C \arrow[ur, "a"] \arrow[rr, "1"'] & {} \arrow[d, phantom, "\Searrow"] & (BA)C \arrow[r, "\rho"] & C(BA) \arrow[ur, "a^*"] \arrow[rr, "1"] & {} \arrow[d, phantom, "\Searrow"] & C(BA) \arrow[ul, "a^*"'] \\
(AB)C \arrow[u, "\rho 1"] \arrow[rr, "1"'] \arrow[dr, "a"'] & {} \arrow[d, phantom, "\Downarrow\varepsilon^{-1}"] & (AB)C \arrow[u, "\rho 1"] \arrow[r, "\rho"'] \arrow[ur, phantom, "\Searrow"] & C(AB) \arrow[u, "1\rho"] \arrow[rr, "1"] \arrow[dr, "a^*"] & {} \arrow[d, phantom, "\Downarrow\eta"] & C(AB) \arrow[u, "1\rho"'] \\
 & A(BC) \arrow[ur, "a^*"'] \arrow[dr, "1\rho"'] \arrow[rrr, phantom, "\Downarrow s"] & & & (CA)B \arrow[ur, "a"'] & \\
 & & A(CB) \arrow[r, "a^*"'] & (AC)B \arrow[ur, "\rho B"'] & &
\end{tikzcd}
$$

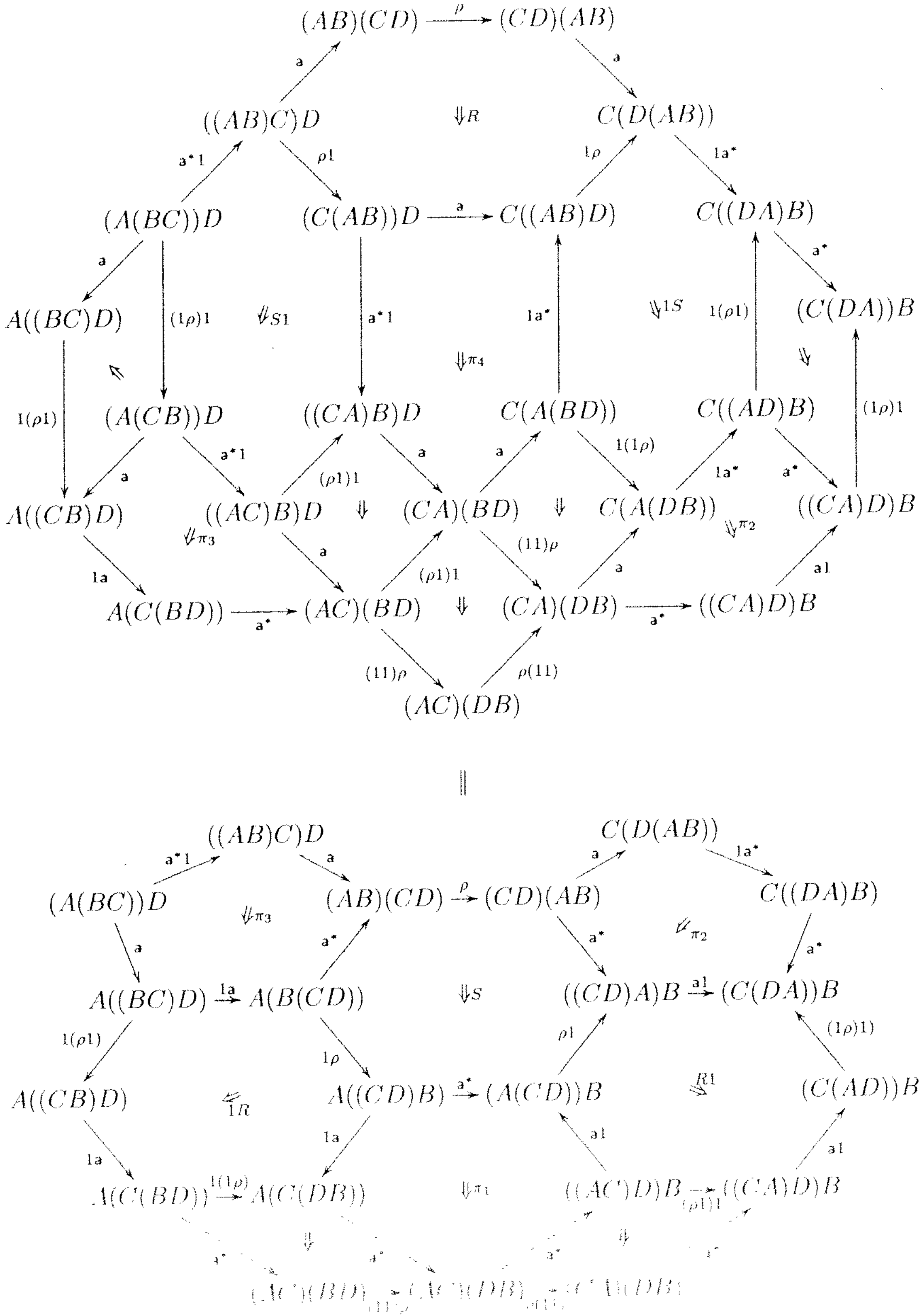
$$\|$$

$$
\begin{tikzcd}
 & & (BA)C \arrow[r, "a"] \arrow[r, phantom, shift right=8ex, "\Downarrow R^{-1}"] & B(AC) \arrow[dr, "1\rho"] & & \\
 & (AB)C \arrow[ur, "\rho 1"] \arrow[dl, "a"'] \arrow[dr, "a"] \arrow[d, phantom, "\Downarrow"] & & & B(CA) \arrow[dr, "a^*"] \arrow[d, phantom, "\Downarrow\varepsilon"] & \\
A(BC) \arrow[rr, "1"] \arrow[d, "1\rho"'] & {} \arrow[d, phantom, "\Swarrow"] & A(BC) \arrow[r, "\rho"'] \arrow[d, "1\rho"] \arrow[dr, phantom, "\Swarrow"] & (BC)A \arrow[ur, "a"] \arrow[rr, "1"'] \arrow[d, "\rho 1"] & {} \arrow[d, phantom, "\Swarrow"] & (BC)A \arrow[d, "\rho 1"] \\
A(CB) \arrow[rr, "1"'] \arrow[dr, "a^*"'] & {} \arrow[d, phantom, "\Downarrow\eta"] & A(CB) \arrow[r, "\rho"] & (CB)A \arrow[rr, "1"] \arrow[dr, "a"'] & {} \arrow[d, phantom, "\Downarrow\varepsilon^{-1}"] & (CB)A \\
 & (AC)B \arrow[ur, "a"] \arrow[dr, "\rho 1"'] \arrow[rrr, phantom, "\Downarrow R"] & & & C(BA) \arrow[ur, "a^*"'] & \\
 & & (CA)B \arrow[r, "a"'] & C(AB) \arrow[ur, "1\rho"'] & &
\end{tikzcd}
$$

A *braided monoidal bicategory* is a monoidal bicategory equipped with a braiding.

We now proceed to the definition of a braided weak monoidal homomorphism.

(BA3) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;



(BA2) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}
 & & & A((BC)D) \arrow[drr, "a^*"] & & & \\
 & A(B(CD)) \arrow[urr, "1a^*"] \arrow[dr, "a^*"] \arrow[ddl, "1(1\rho)"'] & & \Downarrow\pi^* & & (A(BC))D \arrow[dl, "a^*1"'] \arrow[ddr, "\rho"] & \\
 & \Leftarrow & (AB)(CD) \arrow[rr, "a^*"] \arrow[d, "(11)\rho"'] & & ((AB)C)D \arrow[d, "\rho"] & \Leftarrow & \\
A(B(DC)) \arrow[rr, "a^*"] \arrow[dd, "1a^*"'] & & (AB)(DC) \arrow[d, "a^*"] & \Leftarrow_S & D((AB)C) \arrow[d, "a^*"'] & & D(A(BC)) \arrow[ll, "1a^*"'] \arrow[dd, "a^*"] \\
 & \pi^{*-1}\Leftarrow & ((AB)D)C \arrow[rr, "\rho 1"'] & & (D(AB))C \arrow[dd, "a^*1"] & \Leftarrow_{\pi^*} & \\
A((BD)C) \arrow[drr, "a^*"'] & & & & & & (DA)(BC) \arrow[dll, "a^*"] \\
 & & (A(BD))C \arrow[uu, "a^*1"] \arrow[d, "(1\rho)1"'] & \Leftarrow_{S1} & ((DA)B)C & & \\
 & & (A(DB))C \arrow[rr, "a^*1"'] & & ((AD)B)C \arrow[u, "(\rho1)1"'] & &
\end{tikzcd}
$$

$$\|$$

$$
\begin{tikzcd}
 & A(B(CD)) \arrow[dl, "1(1\rho)"'] \arrow[dr, "1a^*"] & & (A(BC))D \arrow[dr, "\rho"] & \\
A(B(DC)) \arrow[dd, "1a^*"'] & & A((BC)D) \arrow[ur, "a"] \arrow[dd, "1\rho"'] & & D(A(BC)) \arrow[dd, "a^*"] \\
 & \Leftarrow_S & & \Leftarrow_S & \\
A((BD)C) \arrow[dr, "1(\rho1)"'] \arrow[dd, "a^*"'] & & A(D(BC)) \arrow[dl, "1a^*"] \arrow[dr, "a^*"] & & (DA)(BC) \arrow[dd, "a^*"] \\
 & A((DB)C) \arrow[dd, "a^*"'] & & (AD)(BC) \arrow[ur, "\rho(11)"] \arrow[dd, "a^*"'] & \\
(A(BD))C \arrow[dr, "1(\rho1)"'] \arrow[r, phantom, "\Leftarrow"] & & \Leftarrow_{\pi^*} 1 & & ((DA)B)C \arrow[l, phantom, "\Leftarrow"] \\
 & (A(DB))C \arrow[rr, "a^*1"'] & & ((AD)B)C \arrow[ur, "(\rho1)1"'] &
\end{tikzcd}
$$

(BA1) for all objects A, B, C and D of $\mathcal{K}$ the following equation holds;

$$
\begin{tikzcd}[column sep=small]
 & & & (A(BC))D \arrow[drr, "a"] & & & \\
 & ((AB)C)D \arrow[urr, "a1"] \arrow[dr, "a"] \arrow[ddl, "(\rho 1)1"'] \arrow[rrrr, phantom, "\Downarrow\pi"] \arrow[dd, phantom, "\Leftarrow"] & & & & A((BC)D) \arrow[dl, "1a"'] \arrow[ddr, "\rho"] \arrow[dd, phantom, "\Leftarrow"] & \\
 & & (AB)(CD) \arrow[rr, "a"] \arrow[dl, "\rho(11)"'] & & A(B(CD)) \arrow[dr, "\rho"] & & \\
((BA)C)D \arrow[r, "a"] \arrow[dd, "a1"'] & (BA)(CD) \arrow[dr, "a"] \arrow[rrrr, phantom, "\Leftarrow_R"] \arrow[dd, phantom, "\Leftarrow_{\pi^{-1}}"] & & & & (B(CD))A \arrow[dl, "a"'] \arrow[dd, phantom, "\Leftarrow_\pi"] & ((BC)D)A \arrow[l, "a1"'] \arrow[dd, "a"] \\
 & & B(A(CD)) \arrow[rr, "1\rho"'] & & B((CD)A) \arrow[ddr, "1a"] & & \\
(B(AC))D \arrow[dr, "a"'] & & & & & & (BC)(DA) \arrow[dl, "a"] \\
 & B((AC)D) \arrow[uur, "1a"] \arrow[dr, "1(\rho 1)"'] \arrow[rrrr, phantom, "\Leftarrow_{1R}"] & & & & B(C(DA)) & \\
 & & B((CA)D) \arrow[rr, "1a"'] & & B(C(AD)) \arrow[ur, "1(1\rho)"'] & &
\end{tikzcd}
$$

$$\Vert$$

$$
\begin{tikzcd}[column sep=small]
 & ((AB)C)D \arrow[dl, "(\rho 1)1"'] \arrow[dr, "a1"] & & A((BC)D) \arrow[dr, "\rho"] & \\
((BA)C)D \arrow[d, "a1"'] \arrow[drr, phantom, "\Leftarrow_R"] & & (A(BC))D \arrow[ur, "a"] \arrow[d, "\rho 1"] \arrow[drr, phantom, "\Leftarrow_R"] & & ((BC)D)A \arrow[d, "a"] \\
(B(AC))D \arrow[dr, "(1\rho)1"'] \arrow[dd, "a"'] & & ((BC)A)D \arrow[dl, "a1"] \arrow[dr, "a"] & & (BC)(DA) \arrow[dd, "a"] \\
 & (B(CA))D \arrow[dd, "a"] \arrow[ddrr, phantom, "\Leftarrow_{\pi^{-1}}"] & & (BC)(AD) \arrow[ur, "(11)\rho"] \arrow[dd, "a"] \arrow[dr, phantom, "\Downarrow"] & \\
B((AC)D) \arrow[dr, "1(\rho 1)"'] \arrow[ur, phantom, "\Leftarrow"] & & & & B(C(DA)) \\
 & B((CA)D) \arrow[rr, "1a"'] & & B(C(AD)) \arrow[ur, "1(1\rho)"'] &
\end{tikzcd}
$$