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Fully dualizable objects in bicategories

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Motivation

In order to understand the "Cobordism Hypothesis", it's required to understand the notion of "fully dualizable objects" (in more generality it applies for (symm. monoidal) (n,∞)-categories)

Fully dualizability resembles the notion of finiteness for the dimension of a vector space.
For 1 TFT we have seen that $Z(pt)$ is a finite dimensional vector space due to the compatibility btw evaluation and coevaluation map).
For the extended version, one wants to give a "categorical notion" of finite dimensionality.
With some examples in bicategories we illustrate this notion.

Recall Duality in symmetric monoidal categories

Def. We say that $V^\vee$ is a right dual of V if $\exists$ maps $ev_V : V\otimes V^\vee \to 1$ and $coev_V : 1 \to V^\vee \otimes V$ satisfying

$$\begin{tikzcd}
 & V\otimes(V^\vee\otimes V) \arrow[r, "\Phi_{V,V^\vee,V}"] & (V\otimes V^\vee)\otimes V \arrow[dr, "ev_V\otimes id_V"] & \\
V\otimes 1 \arrow[ur, "id\otimes coev_V"] & & \circlearrowright & 1\otimes V \arrow[dl, "\sim"] \\
 & V \arrow[ul, "\sim"] \arrow[r, "id_V"] & V &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (V^\vee\otimes V)\otimes V^\vee \arrow[r, "\Phi^{-1}_{V^\vee,V,V^\vee}"] & V^\vee\otimes(V\otimes V^\vee) \arrow[dr, "id\otimes ev_V"] & \\
1\otimes V^\vee \arrow[ur, "coev_V\otimes V"] & & \circlearrowright & V^\vee\otimes 1 \arrow[dl, "\sim"] \\
 & V^\vee \arrow[ul, "\sim"] \arrow[r, "id_{V^\vee}"] & V^\vee &
\end{tikzcd}$$

For symmetric monoidal categories, a right dual is a left dual, so there is no ambiguity to denote the dual by $\cdot^{\vee}$.

Rmk Finite dimensionality for vector spaces (we saw it in detailed in the case of 1TFT) relies on the existence of a coevaluation map.

If $W, W' \in Ob(Vect)$, $ev_V : V \otimes V^{\vee} \longrightarrow k$ induces a map

$$Hom(W, W' \otimes V) \longrightarrow Hom(W \otimes V^{\vee}, W' \otimes V \otimes V^{\vee}) \longrightarrow Hom(W \otimes V^{\vee}, W')$$

$$\alpha \longmapsto \alpha \otimes id_V$$

$$\gamma \longmapsto \gamma \otimes ev_V$$

The composition is an isomorphism iff V is finite dim. where the inverse map

$$Hom(W \otimes V^{\vee}, W') \longrightarrow Hom(W \otimes V^{\vee} \otimes V, W' \otimes V) \longrightarrow Hom(W, W' \otimes V)$$

is given by composing with $coev_V : k \longrightarrow V \otimes V^{\vee}$.

Rmk: The left/right duals in $\mathcal{C}$ are uniquely defined (up to iso)

Examples: • The dualizable objects in Vect(k) are finite dimensional vector spaces.

- Dualizability in $\mathcal{C}at$ (as a 2-cat)
  - $Ob(\mathcal{C}at)$ = (small) categories
  - 1-$Mor(\mathcal{C}at)$ = functors
  - 2-$Mor(\mathcal{C}at)$ = natural transf.

Adjunction of functors

Def: Let $\mathcal{C}, \mathcal{D} \in Ob(\mathcal{C}at)$, $F : \mathcal{C} \longrightarrow \mathcal{D}$, $G : \mathcal{C} \longrightarrow \mathcal{D}$ functors.

An adjunction btw F and G is a collection of bijections

$$\Phi_{C,D} : Hom_{\mathcal{D}}(F(C), D) \simeq Hom_{\mathcal{D}}(C, G(D)))$$

that is (natural) functorial in $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

F is a left adjoint to G (G is the right adjoint of F)

Ex

$F : Set \longrightarrow Gp$ $\quad X \rightarrow$ Free group gen by elements of X.

$G : Gp \longrightarrow Set$ $\quad \mathcal{G} \rightarrow$ the set $\mathcal{G}$.

- F is a left adjoint of G.

Obs: Given $\Phi_{C,D}$ and F, this determines G up to canonical isomorphism. The same for $\Phi_{C,D}$ and G (relies on Yoneda lemma).

Let $\{\Phi_{C,D}\}_{C\in\mathcal{C}, D\in\mathcal{D}}$ be an adjunction btw F and G.

Consider $D = F(C)$

$$\Phi_{C,D} : \mathrm{Hom}_{\mathcal{D}}(D, D) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(C, G(D))$$

$$id_D \longmapsto u_C$$

$u_C : C \longrightarrow (G\circ F)(C)$ (canonical) depends naturally on C.

$\{u_C\}_{C\in\mathcal{C}}$ can be understood as a nat. transformation of the functors

$$u : id_{\mathcal{C}} \longrightarrow G\circ F.$$

Def u is called the unit for the adjunction btw F and G.

Obs. If $u : id_{\mathcal{C}} \longrightarrow G\circ F$ is a natural transf. it gives rise to a can. map

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(F(C), D) \arrow[r] \arrow[rr, bend right, "\Phi"'] & \mathrm{Hom}_{\mathcal{C}}((G\circ F)(C), G(D)) \arrow[r, "\circ u_C"] & \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \end{tikzcd}$$

if Φ is a bijection then we have an adjunction btw F and G.

Def (dual of the unit) $F : \mathcal{C} \longrightarrow \mathcal{D}$
$G : \mathcal{D} \longrightarrow \mathcal{C}$.

If v $F\circ G \longrightarrow id_{\mathcal{D}}$ is a natural transf, it induces

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(C, G(D)) \arrow[r] \arrow[rr, bend right] & \mathrm{Hom}_{\mathcal{D}}(F(C), (F\circ G)(D)) \arrow[r, "v_D \circ"] & \mathrm{Hom}_{\mathcal{D}}(F(C), D) \end{tikzcd}$$

if $\Psi_{C,D}$ is bijective for $C\in\mathcal{C}$, $D\in\mathcal{C}$ $^{\Psi}$ it gives an adjunction betw F and G and v $F\circ G \longrightarrow id_{\mathcal{D}}$ is a counit for the adjunction btw F and G

Compatibility btw adjunction, unit and counit

- $F = \underbrace{F\circ id_{\mathcal{C}} \xrightarrow{id\times u} F\circ G\circ F \xrightarrow{v\times id} id_{\mathcal{D}}\circ F}_{} = F$ $\Big\}$ $(*)$
- $G = id_{\mathcal{C}}\circ G \xrightarrow{u\times id} \overset{id_F}{G\circ F\circ G} \xrightarrow{id\times v} G\circ id_{\mathcal{D}} = G$

If we have nat. transformations $u: id_{\mathcal{C}} \longrightarrow G\circ F$, $v: F\circ G \longrightarrow id_{\mathcal{D}}$ compatible as in $(*)$ then the induced maps

$$\alpha: Hom_{\mathcal{D}}(F(C), D) \longrightarrow Hom_{\mathcal{C}}(C, G(D))$$
$$\beta: Hom(C, G(D)) \longrightarrow Hom(F(C), D))$$

are inverse to each other.

- Left-right adjoints in 2-categories

Def: Let $\mathcal{E}$ be a 2-cat, $X, Y \in Ob(\mathcal{E})$, $f: X \to Y$, $g: Y \to X \in$ 1-Mor$(\mathcal{E})$. A 2-morphism $u: id_X \longrightarrow g\circ f$ is the unit of an adjunction btw f and g if $\exists$ a 2-mor $v: f\circ g \longrightarrow id_Y$ such that

$$\begin{tikzcd} f \simeq f\circ id_X \arrow[r, "id\times u"] \arrow[rr, bend right, "id_f"'] & f\circ g\circ f \arrow[r, "v\times id"] & id_Y\circ f \simeq f \end{tikzcd}$$

$$\begin{tikzcd} g \simeq id_{\mathcal{C}}\circ g \arrow[r, "u\times id"] \arrow[rr, bend right, "id_g"'] & g\circ f\circ g \arrow[r, "id\times v"] & g\circ id_{\mathcal{D}} \simeq g \end{tikzcd}$$

v is called the counit of the adjunction

(or ~~f is~~ a u (or v) exhibit f as a left adjoint of g or g as a right adjoint of f ?

Example: 2-categories vs symm. monoidal categories

Given a symm. monoidal cat $\mathcal{C}$ one can construct a bicategory $B\mathcal{C}$ given by

- $ob(B\mathcal{C}) = *$
- 1-mor$(B\mathcal{C})$: $Map_{B\mathcal{C}}(*,*) = \mathcal{C}$
- 2-morphism (composition) $Map_{B\mathcal{C}}(*,*) \times Map_{B\mathcal{C}}(*,*) \longrightarrow Map_{B\mathcal{C}}(*,*)$

$$(X, Y) \longmapsto X\otimes Y$$

If $\mathcal{E}$ is a 2-cat with a distinguished obj $*$

$\mathcal{C} = Map_{\mathcal{E}}(*,*)$ is a monoidal category.

Using this correspondence.

- X is (right) dual to Y in $\mathcal{C}$ (monoidal category) iff X is a(right) adjoint to Y (viewed as 1-morph) in $B\mathcal{C}$ (using the new notion of "dua

Rmks $f: X \longrightarrow Y$, $g: Y \longrightarrow X$, $u: id_X \longrightarrow g\circ f$

1-mor 1-mor 2-mor

uniquely determine $v: f\circ g \longrightarrow id_Y$ (unit for an adjunction btw f and g)

(a counit for an adjunction) btw f and g

Proof We ~~define~~ say that $v: f\circ g \longrightarrow id_Y$ is upper compatible if

$$\begin{tikzcd} f \simeq f\circ id_X \arrow[r, "id \times u"] \arrow[rr, bend right, "id_f"'] & f\circ g\circ f \arrow[r, "v\circ id"] & id_Y\circ f \simeq f \end{tikzcd}$$

and lower compatible

We can prove the following Lemma:

Lemma: Let $f: X \longrightarrow Y$, $g: Y \longrightarrow X$ 1-mor., $u: id_X \longrightarrow g\circ f$ 2-mor. Supp that $\exists$ $v: f\circ g \longrightarrow id_Y$ upper compatible, and $v': f\circ g \longrightarrow id_Y$ lower compatible with u. Then $v = v'$ and u is the unit of an adjunction btw f and g (and v is the counit).

Proof:

(Group theoretical analog). If G is an (associative) monoid, and f has a right and left inverse g, g' respect. then

$$g = g(fg') = (gf)g' = g'$$

and f is invert.

→ We define $w: f\circ g \longrightarrow id_Y$ by

$$f\circ g \simeq f\circ id_X\circ g \xrightarrow{u} f\circ(g\circ f)\circ g \overset{\xi}{\simeq} (f\circ g)\circ(f\circ g) \xrightarrow{(u\times v')} id_Y\circ id_Y \simeq id_Y$$

the last 2-mor can be factored as

$$(f\circ g)\circ(f\circ g) \xrightarrow{v\times id} id_Y\circ(f\circ g) \xrightarrow{v'} id_Y$$

and then $w = v'\circ(w'\times id_g)$ where

w' is

$$w': f \simeq f\circ id_X \xrightarrow{id_X\times u} f\circ g\circ f \xrightarrow{v\times id} id_Y\circ f \simeq f$$

since ν is upper compatible $\Rightarrow$ $\omega' = id_f$ $\Rightarrow$ $\omega = \nu'$.

Analogously $\omega = \nu \Rightarrow \nu = \nu'$

Example: $f : X \longrightarrow Y$ invertible 1-morph in a 2-cat $\mathcal{E}$
$g : Y \longrightarrow X$ inverse.

There are isomorphisms

$$id_X \simeq g \circ f \qquad f \circ g \simeq id_Y$$

which are the unit (counit) of an adjunction between X and Y.

This leads to a

Proposition: let $\mathcal{E}$ be a 2-cat. with invertible 2-morphisms.
Let $f \in$ 1-mor $(\mathcal{E})$. TFAE

- f is invertible
- f admits a left adjoint
- f " " a right " ".

Definition: A 2-cat $\mathcal{E}$ has adjoints for 1-morphisms if

(1) $\forall$ 1-morphism $f : X \longrightarrow Y$, $\exists$ 1-mor $g : Y \rightarrow X$ and a 2-mor $u : id_X \longrightarrow g \circ f$ which is the unit of an adjunction.

(2) $\forall$ 1-morphism $g : Y \rightarrow X$, $\exists$ 1-morphism $f : X \longrightarrow Y$ and a 2-mor $u : id_X \longrightarrow g \circ f$ which is the unit of an adjunction.

Example $\boxed{Vect(k)}$

A bilinear map $e : V \otimes W \longrightarrow k$ is an evaluation of a duality iff e is a perfect pairing btw V, W i.e V, W are finite dim. and e induces $\varphi_e : V \longrightarrow W^*$.

Definition (Fully dualizability for symm. monoidal bicategories).

We adapt the notion of fully dualizable objects given by Lurie (in general it extends to (n,∞)-categories) for the particular case of symm. monoidal symm. monoidal bi-categories. Fully dualizability plays an important rolle in the Cobordism Hypothesis and in particular the presentation of $2\,Cob^{ext}$ in terms of generators and relations.

Definition (taken from Claudio Sibilia's talk)

Df: A S.M.B consist of a bicategory M together with the following data:

a) $1 \in M$ (a distinguished object)

b) a homomorphism

$$\otimes = (\otimes, \phi^{\otimes}_{(f,f'),(g,g')}, \phi^{\otimes}_{(a,a')}) : M \times M \longrightarrow M$$

c) trasformations

$$a = (a_{abc}, a_{fgh}) : (a \otimes b) \otimes c \longrightarrow a \otimes (b \otimes c)$$
$$l = (l_a, l_f) : 1 \otimes a \longrightarrow a$$
$$r = (r_a, r_f) : a \longrightarrow a \otimes 1$$
$$\beta = (\beta_{ab}, \beta_{fg}) : a \otimes b \longrightarrow b \otimes a$$

For wich there exist an inverse.

d) invertible modifications:

- π, μ, λ, ρ are in P_1, T_1, T_2, T_3
- R, S are in $B_1 B_2$
- σ are in B_3

s.t they satisfy:
- (TA1), (TA2), (TA3) for $(M, 1, \otimes, a, l, r, \pi, \mu, \lambda, \rho)$
- (BA1), (BA2), (BA3), (BA4) " (β, R, S)
- (SA1), (SA2) " " and σ
- (SMA)

Examples

- Cat
- Alg^2 with bimodules

$Ob(Alg^2(k))$ = k-Algebras

1-mor (Alg^2) = A-B modules $\overset{\frown}{A\ M}\overset{\frown}{\ B}$

2-mor (Alg^2) = bimodule homomorphism.

Def: A fully dualizable object in a (symmetric) monoidal bicategory $(\mathcal{E}, \otimes)$ is an element $X \in Ob(\mathcal{E})$ such that $\exists\ X^\vee \in Ob(\mathcal{E})$ and $ev_X : X \otimes X^\vee \longrightarrow 1$, $coev_X : 1 \longrightarrow X \otimes X^\vee$ such that both admit djoints (as 1-morphisms).

~~Ex: In Vect(k) regarded as a 2-cat over a point, fully dualiza-ble objects are finite dimensional vector spaces~~

Non trivial example: Fully dualizable objects in $Alg^2(k)$

Def: The fully dualizable objects in $Alg^2(k)$ are

- Separable
- finitely generated and projective as a k-module

k-algebras.

Rephrasing: F-D objects in $Alg^2(k)$ are separable Frobenius algebra, due to the following

Prop: (see Schommer-Pries)

- (A, λ, e) is a Frobenius algebra.
  (e is an A-central element, $\lambda : A \longrightarrow k$ is a trace and the Frobenius condition is satisfied

$$\sum_i \lambda(x_i) y_i = \sum_i x_i \lambda(y_i) = 1_A$$

  with $e = \sum x_i \otimes y_i \in A \otimes_k A$.

- (A, b) $b : A \otimes A \longrightarrow k$ non-degenerate bilinear form s.t
  $b(xy, z) = b(x, yz)$ and A is a finitely generated projective k-module.

Def (Separability). A k-algebra is separable if $\exists$ an A-central element e. s.t the following holds

$$\sum_i x_i y_i = 1_A . \text{ for } e = \sum_i x_i \otimes y_i .$$

Prop Let A be a k-algebra, $A^e = A \otimes_k A^{op}$. $\mu : A^e \longrightarrow A$ be the
olication map (regarded as an A^e-module map). $J := \ker(\mu) =$

$1 \otimes a \rangle, a \in A.$

TFAE

- A is separable
- ~~The~~ $\exists\, \tilde{e}$ s.t $J\tilde{e}=0$, $\mu(\tilde{e})=1$
- A is projective as a left A^e-module

If k is a field

- A is finite dimensional and classically separable, i.e. $\forall$ field extension K of k, $A\otimes_k K$ is semisimple

Rmk: For k a perfect field (f.e char(k) = 0, or finite fields) every f.d k-Algebra is separables. Hence in Alg(k) with k-perfect the fully dualizable objects are Frobenius algebras.

References

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