Model categories

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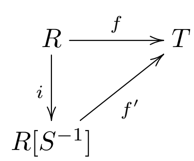
Localization

Motivations for model categories arise from various fields of study, e.g. ring theory and homotopy theory, as we will see. More precisely, what we want to do is to reverse some arrows in a category.

Example 1 (ring theory)**.** Let R be a commutative ring with unity, $S \subset R$ a subset of R (not necessarily a subring). We want to build a ring, usually denoted by $R[S^{-1}]$, where for all $s \in S$ there is an inverse element s^{-1} to s. This process is well known in algebra under the name of localization.
The inclusion $i : R \to R[S^{-1}]$ has the following universal property: for every ring homomorphism $f : R \to T$ such that for every $s \in S$ we have that $f(s) \in T^{\times}$, there is a unique ring homomorphism $f' : R[S^{-1}] \to T$ such that the diagram:



commutes.
We note that R can be described as a category with only one element, where the elements of R are given by the morphisms and multiplication of elements is the composition of arrows (naturally there is the additional structure of an abelian group on the morphisms, thus this is actually a so called category enriched over **Ab**). Then it is evident that the localization of R in S is the process of adding to the category inverses for the arrows corresponding to the elements of S.

Example 2 (homotopy theory)**.** Let **Top** be the category of topological spaces and continuous maps and let W be the subset of the arrows in **Top** given by all weak homotopy equivalences. In homotopy theory we are interested in the study of the distinct weak homotopy types, where two objects are said to have the same weak homotopy type if there is a sequence of weak homotopies joining them. The problem is that weak homotopy equivalence is not an equivalence relation. Let for example K denote the Cantor set with the subspace topology of $\mathbb{R}$ and $K_{\mathrm{d}is}$ be the Cantor set with the discrete topology. Then the map from $K_{\mathrm{d}is}$ to K given by $x \mapsto x$ is a weak homotopy, but it is easy to see that there can be no weak homotopy from K to $K_{\mathrm{d}is}$. Thus we want to make the weak homotopies into isomorphisms to get a real equivalence relation, obtaining $\mathbf{Top}[W^{-1}]$, the so called category of weak homotopy types.

We generalize these concepts as follows:

Definition 3 (Localization)**.** Let $\mathcal{C}$ be a category, S a subset of the arrows of $\mathcal{C}$. The *localization of* $\mathcal{C}$ *in* S is a functor i from $\mathcal{C}$ to a category $\mathcal{C}\ [S^{-1}]$ with the same object set such that $i(c) = c$ for every object $c \in \mathcal{C}$ and that for every functor F from $\mathcal{C}$ to some category $\mathcal{D}$ with the property that for all $s \in S$, $F(s)$ is an isomorphism, there is a unique functor $F' : \mathcal{C}\ [S^{-1}] \to \mathcal{D}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle i}\downarrow & \nearrow_{F'} & \\ \mathcal{C}\ [S^{-1}] & & \end{array}$$

Remark 4. It is important to note that the localization of a category might not have sets for hom-sets, but only classes instead. We will not elaborate on the case.

An application of this is obviously the localization in a commutative ring. here we give two other simple examples:

Example 5. $\mathcal{C}$ is the category with two elements and only one arrow a between them. We write $\mathcal{C} = 0 \xrightarrow{a} 1$. Then, with the same notation:

$$\mathcal{C}\ [\{a\}^{-1}] = 0 \underset{a^{-1}}{\overset{a}{\rightleftarrows}} 1$$

with $a^{-1} \circ a = \mathrm{id}_0$ and $a \circ a^{-1} = \mathrm{id}_1$.

This was an example of a very simple localization. With more complicated categories it becomes immediately very difficult to describe the arrows for the localization.

Example 6. Let $\mathcal{C}$ be the following category:

$$\mathcal{C} = 0 \xleftarrow{a} 1 \xrightarrow{b} 2 \xleftarrow{c} 3$$

We want to find $\mathcal{C}\ [\{b\}^{-1}]$. Unfortunately, the following is not enough:

$$\mathcal{C}\ [\{b\}^{-1}] = 0 \xleftarrow{a} 1 \underset{b^{-1}}{\overset{b}{\rightleftarrows}} 2 \xleftarrow{c} 3$$

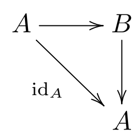
since this construct is no more a category, missing the various compositions. It is in fact necessary to add three more arrows (the various compositions of arrows with b^{-1}) in order to obtain the localization of $\mathcal{C}$ in $\{b\}$.

What we want now is some special kind of categories where, given such a category $\mathcal{C}$ and a some kind of subset of its arrows W, it is possible to describe exactly what the sets of homomorphisms in the localization $\mathcal{C}\ [W^{-1}]$ look like. That is where model categories come into play.

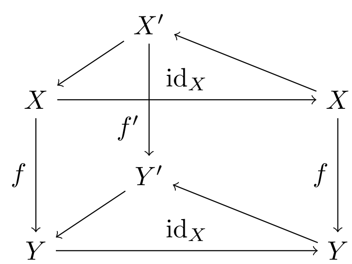
Preliminary notions

We recall some notions from category theory that we will need later on.

Definition 7. Let A and B be objects of a category $\mathcal{C}$. We say that A is an *object retract* of B if there are two arrows, one from A to B and one from B to A, such that the following diagram commutes:

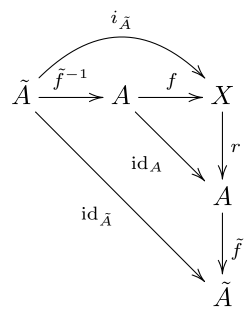


Let $f : X \to Y$ and $f' : X' \to Y'$ be arrows in a category $\mathcal{C}$. Then f is said to be an *arrow retract* (or simply a *retract*) of f' if it is an object retract of f' in the category of arrows on $\mathcal{C}$, i.e. the category with arrows and commutative squares as objects and morphisms respectively. Equivalently, f is a retract of f' if there are arrows such that the following diagram commutes:



Example 8. We can easily see that the notion of an object retract is a natural generalization of the topological concept of a retract.
Indeed, let X be some topological space, A a retract of X and $r : X \to A$ the retraction. Then r is a left inverse for the inclusion map of A in X, i_A.
Conversely, let X and A be two objects of the category **Top** of topological spaces and continuous maps, and $r : X \to A$ a retraction of some function $f : A \to X$ in the sense defined above. Then we have that f is surely injective (else $r \circ f \neq \mathrm{id}_A$) and thus bijective on its image $\tilde{A} = f(A) \subset X$. Let $\tilde{f}$ be f with its codomain restricted to $\tilde{A}$. Obviously, $r|_{\tilde{A}} = \tilde{r}$ is also a bijection. $\tilde{f}^{-1}$ is a continuous function, since it is a well defined function and it equals $r \circ \mathrm{id}_A$. Thus $\tilde{A} \cong A$. This way we get that $\tilde{A}$ is a retract of X (in the topological sense)

by the following commutative diagram:



where $i_{\tilde{A}}$ is the inclusion of $\tilde{A}$ in X. Notice that the retraction for $\tilde{A}$ is given by $\tilde{f} \circ r$.

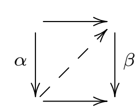
Definition 9. Let $\mathcal{C}$ be a category. An object t is *terminal* in $\mathcal{C}$ if for every object $c \in \mathcal{C}$ there is exactly one arrow $c \to t$. An object s is *initial* in $\mathcal{C}$ if for every object $c \in \mathcal{C}$ there is exactly one arrow $s \to c$.

Remark 10. Terminal and initial objects are unique up to isomorphism (as you can readily check).

Example 11. Let **Top** be the category of topological spaces and continuous functions. Then the empty set $\emptyset$ is an initial object of **Top** and the one point set $*$ is a terminal object.

Model Categories

Definition 12. Let α and β be two morphisms in a category $\mathcal{C}$. We say that α has *left lifting property* with respect to β, and that β has *right lifting property* with respect to α, if for every commuting square there is a dashed arrow such that the following diagram commutes:



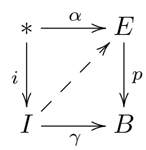
We denote this by $\alpha \pitchfork \beta$.
Let S, T be two subsets of the arrows of $\mathcal{C}$. We say that S has the left lifting property with respect to T, and that T has the right lifting property with respect to S, if for every $\alpha \in S$, $\beta \in T$ we have $\alpha \pitchfork \beta$. In this case we write $S \pitchfork T$. If S is any subset of the arrows of $\mathcal{C}$, we define the following two other subsets of the arrows of $\mathcal{C}$:

$$S^{\pitchfork} = \{\beta | s \pitchfork \beta,\ \forall s \in S\}$$

$$^{\pitchfork}S = \{\alpha | \alpha \pitchfork s,\ \forall s \in S\}$$

We give two examples of lifting properties or object defined trough some lifting property. Both come from topology.

Example 13. Let B be a topological space, E a covering space of B with covering map $p : E \to B$. Then the map $i : * \to I = [0, 1]$ sending the one point set to the element $\{0\} \in I$ has left lifting property with respect to p. Indeed, take $\gamma : I \to B$ any path and $\alpha : * \to E$ sending $*$ to any point in $p^{-1}(\gamma(0))$. Then there is exactly one dashed arrow making the following diagram commute:

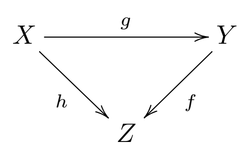


Example 14. A continuous map $p : X \to Y$ between two topological spaces X and Y is called a Serre fibration if it has the right lifting property with respect to all inclusion maps in the set $\{i : Z \times \{0\} \to Z \times I | Z \text{ is a CW} - \text{complex}\}$.

Definition 15. A *closed model category* is a tuple $(\mathcal{M}, W, C, F)$, where $\mathcal{M}$ is a category and W, C and F are three classes of functions, called weak equivalences, cofibrations and fibrations respectively, such that the following axioms hold:

CM1 $\mathcal{M}$ is closed under limits (complete) and colimits (cocomplete).

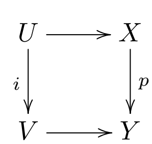
CM2 Let X, Y, Z be objects of $\mathcal{M}$ and f, g, h be morphisms such that the following diagram commutes:



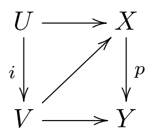
If two of f, g, h are weak equivalences, then so is the third.

CM3 Let f, g be morphisms in $\mathcal{M}$. If f is a retract of g and g is a weak equivalence, fibration or cofibration, then so is f.

CM4 Assume the following diagram commutes in $\mathcal{M}$:



If i is a cofibration and p a fibration, and one of the two is trivial (i.e. it is also a weak equivalence) then there is an arrow from V to X such that the following diagram commutes:



Said in another way, $C \pitchfork (F \cap W)$ and $(C \cap W) \pitchfork F$.

CM5 Let X, Y be objects in $\mathcal{M}$ and f a morphism from X to Y. Then f can be factored in the two following ways:

(a) $f = p \circ i$, where p is a fibration and i is a trivial cofibration (i.e. a cofibration that is at the same time a weak equivalence).

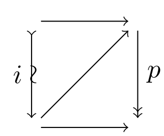
(b) $f = q \circ j$, where q is a trivial fibration (at the same time fibration and weak equivalence) and j is a cofibration.

Remark 16. The original definition, given by Quillen in [3.], is slightly different. The definition we use is more refined and powerful for our needs.

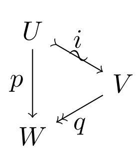
To simplify the reading of the diagrams, we will from now on use $\longrightarrow\!\!\!\!\to$ for fibrations, $\rightarrowtail$ for cofibrations and put a little $\sim$ on weak equivalences.

Lemma 17. Let $(\mathcal{M}, W, C, F)$ be a model category. Then $(C \cap W)^{\pitchfork} = F$ (i.e. a map is a fibration if and only if it has the right lifting property with respect to all trivial cofibrations), $C^{\pitchfork} = F \cap W$, $C \cap W = {}^{\pitchfork}F$ and $C = {}^{\pitchfork}(F \cap W)$.

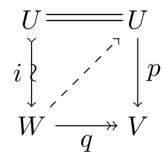
Proof. We will prove only the first statement. The other are proven in a similar way.
Assume p is a fibration and i any trivial cofibration. Then, by **CM4**, there a commutative diagram as follows:



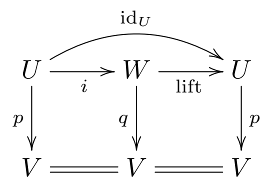
Thus $(C \cap W) \pitchfork p$.
Conversely, assume $(C \cap W) \pitchfork p$. By **CM5**, there are a trivial cofibration i and a fibration q such that $p = q \circ i$, or, seen as a diagram:



Then, by right lifting property of p with respect to trivial cofibrations, we have the following dotted arrow:



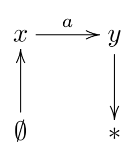
Thus, p is a retract of q, as the following diagram shows:



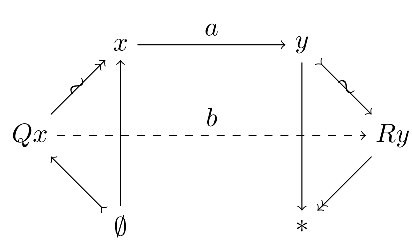
Then by **CM3**, p is a fibration. □

Remark 18. What just proven is what makes a model category "closed". As already said, the original axioms of Quillen for a model category were different, and didn't imply this. It was noted, though, that the vast majority of examples were of closed model categories, and not simply of model categories, thus the axioms were changed to the ones we gave. From now on we will drop the adjective "closed", when speaking of model categories, because nowadays it is only a historical artefact.

If $\mathcal{M}$ is a model category, the fact that it is complete and cocomplete (**CM1**) implies that it has some initial object $\emptyset$ and some terminal object $*$. Now let x and y be objects of $\mathcal{M}$, and $a \in \hom_{\mathcal{M}}(x, y)$. Then we have:



Where the arrows from $\emptyset$ and to $*$ are unique. Then by **CM5** we have the following factorization:



This implies that in the localization of $\mathcal{M}$ in W (the set of weak equivalences) we can interchange a and b (since weak equivalences become isomorphisms). Objects like Qx and Ry will have a lot of importance later on, and thus we give them a name:

Definition 19. Let $\mathcal{M}$ be a model category with initial object $\emptyset$ and terminal object $*$. An object y of $\mathcal{M}$ is called *fibrant* if the unique map $y \to *$ is a fibration. An object x of $\mathcal{M}$ is called *cofibrant* if the unique map $\emptyset \to x$ is a cofibration.

What we have done above can thus be interpreted as the fact that we can always change x and y with some cofibrant and fibrant objects Qx and Ry.

Remark 20. Qx and Ry are not fix: they depend on a.

Example 21. In the case of the category **Top**, we can take fibrations to be Serre fibrations, cofibrations the continuous functions having the lifting property of **CM4** and weak equivalences to be weak homotopy equivalences. Then **Top** together with those three types of maps is a model category in which every object is fibrant and all CW-complexes are cofibrant.

Definition 22. Let $\mathcal{M}$ be a model category, $f, g : X \to Y$ two arrows in $\mathcal{M}$, $Y \times Y$ the product object of Y and Y, $X \sqcup X$ the coproduct object of X and X. Then:

- Take the map $\Delta : Y \to Y \times Y$ induced by the identity map and factorize it as in **CM5**(a):

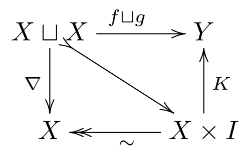
$$\begin{array}{ccc} & & Y \times Y \\ & \overset{\Delta}{\nearrow} & \uparrow \\ Y & \xrightarrow[\sim]{} & Y^I \end{array}$$

  Then Y^I is the so called *path object* of X.
  We say that f is *right homotopic* to g, and write $f \overset{r}{\sim} g$, if there is a map $H : X \to Y^I$ such that the following diagram commutes:

$$\begin{array}{ccc} & & Y \times Y \\ & \overset{f\times g}{\nearrow} & \uparrow \\ X & \xrightarrow[H]{} & Y^I \end{array}$$

  where the map from Y^I to $Y \times Y$ is the one of the previous diagram.

- Similarly, let $\nabla : X \sqcup X \to X$ be the map induced by the identity map. We say that f and g are *left homotopic*, and write $f \overset{l}{\sim} g$, if there is an arrow K such that the following diagram commutes:



where we have used **CM5**(b) to decompose ∇. $X \times I$ is called the *cylinder object* of X.

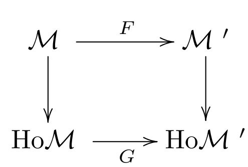
Remark 23. If X is fibrant, then right homotopy is an equivalence relation on the hom-set $\hom(X, Y)$. If Y is cofibrant, then left homotopy is an equivalence relation. If X is fibrant and Y is cofibrant, then $f \overset{r}{\sim} g \Leftrightarrow f \overset{l}{\sim} g$. in this case, we write $f \sim g$.
For a proof of those facts, refer to [4.].

Remark 24. It is quite easy to see both the notions of left/right homotopy equivalence generalize the notion of homotopy of maps in **Top**. First of all we notice that there is in fact an object I in **Top** (given by the closed unit interval) such that $X \times I$ is the cylinder object of X and such that Y^I is the path object of Y. The definition for left homotopy equivalence is then exactly the definition of homotopy. Right homotopy equivalence can be seen to express the same concept by noting that there is a (quite obvious) isomorphism $\hom(X \times I, Y) \cong \hom(X, Y^I)$.

What we do now is to replace the model category $\mathcal{M}$ by its localization in the weak equivalences $\mathcal{M}\,[W^{-1}]$. We will denote this new category by $\mathrm{Ho}\mathcal{M}$ and call it the homotopy category of $\mathcal{M}$. Here all weak equivalences are isomorphisms, and for this reason, with a little work, we can define Q and R as functors from $\mathcal{M}$ to $\mathrm{Ho}\mathcal{M}$, the so called cofibrant and fibrant replacements (see for example [4.]). We can then have a complete representation of $\hom_{\mathrm{Ho}\mathcal{M}}\,(x, y)$ for any two objects x and y. It is given by $\hom_{\mathrm{Ho}\mathcal{M}}\,(x, y) \cong \hom_{\mathcal{M}}\,(Qx, Ry)/\sim$. We will not give proof of this fact, but it is something very useful when trying to describe the homotopy category $\mathrm{Ho}\mathcal{M}$.

Yoga of derived functors

When speaking of categories, functors are of fundamental importance. Thus given two model categories $(\mathcal{M}, W, C, F)$, $(\mathcal{M}', W', C', F')$ and a functor F from $\mathcal{M}$ to $\mathcal{M}$ ', we'd like to derive a functor from $\mathrm{Ho}\mathcal{M}$ to $\mathrm{Ho}\mathcal{M}'$ such that the localization is preserved in some sense. A first, very intuitive approach would be to ask for a functor G making the following diagram commute:



where the vertical arrows are the localizations.

Example 25. Take the functor $\pi_n : \mathbf{Top}_* \to \mathbf{Gp}$, $n \geq 1$. The weak equivalences in $\mathbf{Top}_*$ are the weak homotopy equivalences, in $\mathbf{Gp}$ they are simply the isomorphisms. Then $\mathbf{Gp}$ is the localization of itself and thus we can use the universal property of the localization to the following dashed functor:

$$\begin{array}{ccc} \mathbf{Top}_* & \xrightarrow{\pi_n} & \mathbf{Gp} \\ \downarrow & & \| \\ \mathrm{Ho}\mathbf{Top}_* & \dashrightarrow & \mathrm{Ho}\mathbf{Gp} \end{array}$$

It is quite obvious that a functor satisfying this property can exist only if $F(W) \subset W'$. It turns out that this requirement is too strict.
We will loosen our requirements to accept a broader class of functors. Let $(\mathcal{M}, W, C, F)$ be a model category, $\mathcal{C}$ any category and F a functor from $\mathcal{M}$ to $\mathcal{C}$. Let G be a functor from $\mathrm{Ho}\mathcal{M}$ to $\mathcal{C}$. We require that the following diagram commutes *up to a natural transformation*:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle i}\downarrow & \Uparrow & \nearrow {\scriptstyle G} \\ \mathrm{Ho}\mathcal{M} & & \end{array} \quad \text{or} \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{F} & \mathcal{C} \\ {\scriptstyle i}\downarrow & \Downarrow & \nearrow {\scriptstyle G} \\ \mathrm{Ho}\mathcal{M} & & \end{array}$$

where i denotes the localization. That means that we require that either $F \Rightarrow G \circ i$ or $G \circ i \Rightarrow F$. We also require that G has some kind of uniqueness property, so that it can be defined well, when it exists. To define this we need the following definitions:

Definition 26. Let $(\mathcal{M}, W, C, F)$ be a model category and $\mathcal{C}$ any category. Then we denote by $\mathrm{Fun}(\mathcal{M}, \mathcal{C})$ the set of functors from $\mathcal{M}$ to $\mathcal{C}$ and by $\mathrm{Fun}^W(\mathcal{M}, \mathcal{C})$ the set of functors from $\mathcal{M}$ to $\mathcal{C}$ such that weak equivalences are sent to isomorphisms.

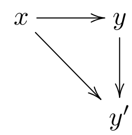
Definition 27. Let $\mathcal{C}$ be a category, D a subset of the objects of $\mathcal{C}$ and x some object of $\mathcal{C}$. Then we define the following two categories:

- D/x is the category with as objects the arrows $y \to x$ with $y \in D$ and as arrows the commutative triangles:

$$\begin{array}{ccc} y & \longrightarrow & x \\ \downarrow & \nearrow & \\ y' & & \end{array}$$

  where $y, y' \in D$.

- $x\backslash D$ is the category with as objects the arrows $x \to y$ with $y \in D$ and as arrows the commutative triangles:



  where $y, y' \in D$.

We notice that in fact $\mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )$ is a subset of the objects of the category of functors, thus the following definition, which gives us the sought uniqueness, makes sense:

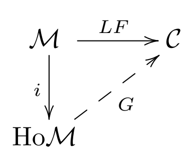
Definition 28. Let $(\mathcal{M}, W, C, F)$ be a model category, $\mathcal{C}$ any category, F a functor from $\mathcal{M}$ to $\mathcal{C}$. Then:

- If the category $\mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )/F$ has a terminal object, we call it the *left derived functor* and denote it by LF. In fact, this object is a natural transformation. With an abuse of notation, we also denote its codomain LF.

- If the category $F\backslash\mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )$ has an initial object, we call it the *right derived functor* and denote it by RF. Again, with an abuse of notation we also denote its domain by RF.

Remark 29. If F sends weak equivalences to isomorphisms, we get that $F = LF = RF$.
For the case of $F = LF$, take the category $\mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )/F$. We notice that $F \Rightarrow F$ is in fact in this category, and thus it is its terminal object. The case for RF is very similar.
In the general case, we can see those functors as the ones that are the closest to F in their respective categories.

We can then use LF, RF to get the functors we wanted. Assume for example that LF exists. Then, since $LF \in \mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )/F$, it is actually a natural transformation $LF \Rightarrow F$ from some functor $LF \in \mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )$ to F. This means, by definition 26, that the functor LF sends weak equivalences to isomorphisms. Thus we can apply the universal property of the localization to get a suitable functor:



We automatically have the natural transformation $G \circ i = LF \Rightarrow F$. Furthermore, since the natural transformation LF is unique up to isomorphism (being a terminal object), we have that the functor LF is also unique up to isomorphism (invertible natural transformation), by the definition of $\mathrm{Fun}^W(\mathcal{M}\ ,\mathcal{C}\ )/F$, and the same is then valid for G, as we wanted.

Main example: cochain complexes

We present now an important example of model category: the (co)chain complexes of left (or right) modules over a ring R.

Definition 30. The category $\mathbf{coCh}(R)$ of cochain complexes of left modules over a ring R is the following category:
Objects are sequences of left R-modules and maps $(M,d) = (M_n, d_n)_{n\in\mathbb{Z}}$ with $d_n : M_n \to M_{n+1}$ such that the composition of two successive such arrows is the trivial map (i.e. $d_n(M_n) \subset \ker(d_{n+1})$). We often represent such an object by:

$$\dots \xrightarrow{d_{-2}} M_{-1} \xrightarrow{d_{-1}} M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} \dots$$

Morphisms between two chains (M,d), (N,e) are sequences of functions $f = (f_n)_{n\in\mathbb{Z}}$, $f_n : M_n \to N_n$ such that every square of the following diagram commutes:

$$\begin{array}{ccccccccccc} \dots & \xrightarrow{d_{-2}} & M_{-1} & \xrightarrow{d_{-1}} & M_0 & \xrightarrow{d_0} & M_1 & \xrightarrow{d_1} & M_2 & \xrightarrow{d_2} & \dots \\ & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \\ \dots & \xrightarrow[e_{-2}]{} & N_{-1} & \xrightarrow[e_{-1}]{} & N_0 & \xrightarrow[e_0]{} & N_1 & \xrightarrow[e_1]{} & N_2 & \xrightarrow[e_2]{} & \dots \end{array}$$

In other words, $f_{n+1} \circ d_n = e_n \circ f_n$.

Definition 31. We define two classes of objects in $\mathbf{coCh}(R)$. Let M be a left R-module, then:

- $S^n(M)$ is the chain with M at index n and the zero module at every other index.
- $D^n(M)$ is the chain with M at indexes $n-1$ and n (with identity map between them) and the zero module at every other index.

We usually use the notation $S^n = S^n(R)$ and $D^n = D^n(R)$.

Definition 32. The n^{th} homology for cochain complexes is the functor $H_n : \mathbf{coCh}(R) \to R - \mathbf{mod}$ defined by $H_n(M,d) = \ker(d_n)/d_{n-1}(M_{n-1})$. If $f : (M,d) \to (N,e)$, then $H_n(f) : H_n(M,d) \to H_n(N,e)$ is defined by $f([x]) = [f_n(x)]$.

We now have all we need to construct a model structure on $\mathbf{coCh}(R)$.
First of all we define the class of weak equivalences by

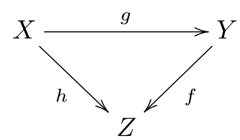
$$W = \{f | H_n(f) \text{ is a bijection } \forall n \in \mathbb{Z}\}$$

Then we take the following two sets of arrows:

- $I = \{S^{n-1} \to D^n\}$
- $J = \{0 \to D^n\}$

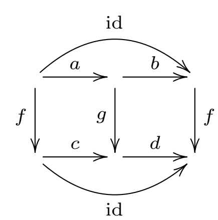
From these we define our set of fibrations as $F = J^{\pitchfork}$, and our set of cofibrations as $C = {}^{\pitchfork}(I^{\pitchfork})$.
We show two of the axioms needed for this to actually be a model structure on $\mathbf{coCh}(R)$:

CM2 Assume that two out of three of g, f and h are weak equivalences and that the following diagram commutes:

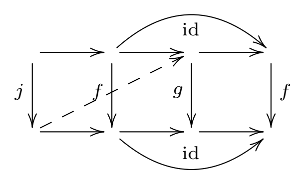


We show that the third arrow is also a weak equivalence.
If f and g are weak equivalences, then by definition we have that for all n, $H_n(f)$ and $H_n(g)$ are bijective. Since H_n is a functor, we have that $H_n(h) = H_n(f) \circ H_n(g)$, and it is thus bijective, proving that h is also a weak equivalence.
Similar reasoning proves the other two cases.

CM3 Let f, g be two arrows, and let f be a retract of g. We have



Assume g is a weak equivalence. Then, since $\mathrm{id} = b \circ a$, we have $H_n(\mathrm{id}) = H_n(b) \circ H_n(a)$ and thus (since $H_n(\mathrm{id})$ is obviously bijective) that $H_n(a)$ is injective and $H_n(b)$ is surjective. Similarly, $H_n(c)$ is injective, $H_n(d)$ is surjective. Note then that $g \circ a = c \circ f$ (since the first square commutes. Since $H_n(g)$ is bijective by assumption and, as we have shown, $H_n(a)$ is injective, we have that $H_n(g) \circ H_n(a)$ is also injective, and thus that $H_n(f)$ also is. Similarly, using the second square, we get that $H_n(f)$ is surjective, and thus that f is a weak equivalence.
Assume that g is a fibration. Then it is easy to see that f is also a fibration (i.e. $J \pitchfork f$). Indeed, let $j \in J$. Then the following diagram shows that f has right lifting property with respect to j:



where the dashed arrow is given by right lifting property of g with respect to the set J.

The case where g is a cofibration is similar.

In the book of Hovey ([4.]), the very similar of the category $\mathbf{Ch}(R)$ of chain complexes over a ring R is treated in more detail.

Remark 33. The category $\mathbf{coCh}(R)$ is in fact a case of what is called a *cofibrantly generated model category*. The general construction of this kind of category is done by choosing a set W of weak equivalences and two suitable generating sets of arrows I and J, with $J \subset I$, and then defining fibrations and cofibrations as follows:

$$F = J^{\pitchfork} \qquad (F \cap W) = I^{\pitchfork}$$

$$(C \cap W) = {}^{\pitchfork}(J^{\pitchfork}) \qquad C = {}^{\pitchfork}(I^{\pitchfork})$$

For J and I satisfying some conditions, this gives in fact a model category.

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