Model structure on topological spaces and on simplicial set

Julian Zimmert

Cofibrantly generated model categories

motivation
It is in general very hard to find or construct a model structure for a given category. The main example to do so are the cofibrantly generated model categories.

Definition 1 (cofibrantly generated model category)**.** *A model category* $(\mathcal{C}, W, C, F)$ *is cofibrantly generated if there are sets* I *and* J *such that*

- $F = J^{\pitchfork}$
- $F \cap W = I^{\pitchfork}$
- *The domains of* I *are small relative to* I*-cell.*
- *The domains of* J *are small relative to* J*-cell.*

I *is called the set of generating cofibrations.* J *is called the set of generating trivial cofibrations.*

So we can use those sets I and J to construct a model structure.

Theorem 2. $\mathcal{C}$ *is a category with all small colimits and limits,* W *is a subcategory and* I, J *are sets of maps. Then* $\mathcal{C}$ *is a cofibrantly generated model category with* I *as the set of generating cofibrations and* J *set of generating trivial cofibrations if and only if*

- W *has the two out of three property and is closed under retracts.*
- $I^{\pitchfork} \subset W \cap J^{\pitchfork}$
- *Either* $W \cap {}^{\pitchfork}(I^{\pitchfork}) \subset {}^{\pitchfork}(J^{\pitchfork})$ *or* $W \cap J^{\pitchfork} \subset I^{\pitchfork}$
- *The domains of* I *are small relative to* I*-cell.*
- *The domains of* J *are small relative to* J*-cell.*
- J*-cell* $\subset W \cap {}^{\pitchfork}(I^{\pitchfork})$

Topological spaces

Let Top denote the category of topological spaces and continuous maps.
D^n is the unit disk in $\mathbb{R}$ and S^{n-1} is the unit sphere.
There is the boundary inclusion $S^{n-1} \to D^n$ and define $S^{-1} = \emptyset$, $D^0 = \{0\}$.

Definition 3. *Take $\star = \{1, 0, 0, ..., 0\}$ as basepoint in S^n. For a given X with basepoint $x \in X$, the set of pointed homotopy classes of pointed maps from $(S^n, \star)$ to (X, x) is $\pi_n(X, x)$.*

Definition 4 (weak equivalences)**.** *A map $f : X \to Y$ in Top is a weak equivalence if*

$$\pi_n(f, x) : \pi_n(X, x) \to \pi_n(Y, f(x))$$

is an isomorphism for all $n \geq 0$ and $x \in X$

Theorem 5 (Model structure of Top)**.** *Top is cofibrantly generated with I and J such that:*
The set of generating cofibrations I is the set of boundary inclusions $S^{n-1} \to D^n$.
The set of generating trivial cofibrations J is the set of trivial inclusions $D^n \to D^n \times [0, 1] : x \to (x, 0)$.

Simplicial sets

Definition 6 (simplicial category)**.** *The simplicial category Δ contains the objects*

$$[n] = \{0, 1, 2, ..., n\}$$

for $n \geq 0$ and $\Delta([n], [k])$ the set of weakly order preserving maps f from $[n]$ to $[k]$. $x \leq y$ implies $f(x) \leq f(y)$

There are two subcategories:

- Δ_+ of injective order preserving maps
- Δ_- of surjective order preserving maps

Each morphism in Δ can be factored uniquely into a morphism in Δ_- followed by a morphism in Δ_+.
Δ is generated by $d^i : [n-1] \to [n] \in \Delta_+$ for $n \geq 1$ and $0 \leq i \leq n$, with the image of d^i not including i and the morphisms $s^i : [n] \to [n-1] \in \Delta_-$ for $n \geq 1$ and $0 \leq i \leq n-1$, where s^i identifies i and $i+1$.

Definition 7 (category of simplicial/cosimplicial objects)**.** *For a category $\mathcal{C}$ the category of functors $\mathcal{C} \to \Delta$ is called the category of cosimplicial objects.*
The functor category of functor $\Delta^{op} \to \mathcal{C}$ is calles the category of simplicial

objects.
If $\mathcal{C}$ is the category of sets we denote $\Delta^{op} \to Set$ as SSet and call it the category of simplicial sets.

Definition 8 (set of n-simplices)**.** *If K is a simplicial set, we call $K_n := K[n]$ the set of n-simplices of K. If $x \in K_n$, the integer n is referred to as the dimension of x.*
Dual to the d^i we have the face maps $d^i : K_n \to K_{n-1}$ for $n \geq 1$ and $0 \leq i \leq n$. Dual to s^i we have the degeneracy maps $s^i : K_{n-1} \to K_n$ for $n \geq 1$ and $0 \leq i \leq n-1$.

Definition 9 (Kan-Complex)**.** *The fibrant objects of SSet are called $Kan - Complexes$.*

Definition 10 ($\Delta[-]$)**.** *The functor $\Delta[-] : \Delta \to SSet$ is defined by the functor $\Delta(-,-) : \Delta^{op} \times \Delta \to Set$.*

So $\Delta[n]$ is the functor that takes $[k]$ to $\Delta([k],[n])$. The simplicial set $\Delta[n]$ has $\binom{n}{k}$ nondegenerate k-simplices, corresponding to the injective order-preserving maps $[k] \to [n]$, and in particular one nondegenerate n-simplex i_n.

Definition 11 ($\delta\Delta[-]$)**.** *$\delta\Delta[n]$ is the boundary of $\Delta[n]$. The nondegenerate k-simplices correspond to nonidentity injective order-preserving maps $[k] \to [n]$.*

Definition 12 ($\Lambda^r[n]$)**.** *For an r with $0 \leq r \leq n$, the simplicial set $\Lambda^r[n]$ has nondegenerate k-simplices all injective order-preserving maps $[k] \to [n]$ except the identity and the injective order-preserving map $[n-1] \xrightarrow{d^r} [n]$ whose image does not contain r.*
It is called the r-horn of $\Delta[n]$.

The simplicial set $\Lambda^r[n]$ is the closed star of the vertex r in $\Delta[n]$.

Definition 13 (geometric realization)**.** *The geometric realization of $\Delta[n]$ is defined as the points $(t_0, ..., t_n) \in R^n$ such that $t_i \geq 0 \forall i$ and $\sum t_i = 1$.*
This definition induces an adjunction $(||, Sing, \varphi) : SSet \to Top$.
The left adjoint $||$ is called the geometric realization.
The right adjoint $Sing$ is called the singular functor.

Theorem 14 (Model Structure in SSet)**.** *SSet is a cofibrantly generated model category with:*
The set I consists of the canonical inclusions $\delta\Delta[n] \to \Delta[n]$ for $n \geq 0$.
The set J consists of the canonical inclusions $\Lambda^r[n] \to \Delta[n]$ for $n > 0$ and $0 \leq r \leq n$.
A map $f \in SSet$ is a weak equivalence if and only if $|f|$ is a weak equivalence in Top.

Quillen Equivalence

Definition 15 (Quillen functor)**.** *For $\mathcal{C}$ and D, two model categories, a left quillen functor $F : \mathcal{C} \to D$ is a left adjoint that preserves cofibrations and trivial cofibrations.*

Definition 16 (Quillen Adjunction)**.** *An adjunction* (F, U, φ) *from* $\mathcal{C}$ *to* D *with* $F : \mathcal{C} \to D$ *the left adjoint,* $U : D \to \mathcal{C}$ *the right adjoint and* $\varphi : D(FA, B) \to \mathcal{C}(A, UB)$ *the natural isomorphism expressing* U *as a right adjoint of* F *is a quillen adjunction if* F *is a left quillen functor.*

Definition 17 (Quillen Equivalence)**.** *A Quillen adjunction* $(F, U, \varphi) : \mathcal{C} \to D$ *is called a Quillen equivalence if and only if, for all cofibrant* X *in* $\mathcal{C}$ *and fibrant* Y *in* D*, a map* $f : FX \to Y$ *is a weak equivalence in* D *if and only if* $\varphi(f) : X \to UY$ *is a weak equivalence in* $\mathcal{C}$ *.*
It is equivalent to define a Quillen equivalence as an Quillen adjunction where the derived adjunction $L(F, U, \varphi)$ *is an adjoint equivalence of categories.*

Remark 18. *The category SSet is Quillen equivalent to Top with the adjuntion* $(||, Sing, \varphi)$