AN OUTLINE OF THE THEORY OF $(\infty, 1)$-CATEGORIES

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Abstract. This exposition offers an overview of the theory of $(\infty, 1)$-categories and aims at providing a starting point for a more detailed study. Our main focus is on discussing the most popular definitions of $(\infty, 1)$-categories, namely simplicial categories, relative categories, complete Segal spaces, Segal categories and quasi-categories as well as the relationships between each of these models. It is impossible to give a self contained account of such a vast subject in so little space so we have included exhaustive references in order to facilitate filling in details as the reader sees fit. The only other exposition of similar scope that we know of is [Cam13].

Contents

1. Introduction

This exposition is organised into two parts. The first part consists of §2 in which we discuss

- why higher categories and especially "weak" higher categories are interesting,
- the properties we may expect that they satisfy,
- the difficulties in providing a definition similar to common ones of ordinary categories and
- a different approach to categories circumventing these difficulties.

The second part comprises §§3 - 6.

- We apply the considerations of the first part to give a definition of $(\infty, 0)$-categories in §3 as well as several definitions of $(\infty, 1)$-categories in §4 and
- we then prove in §5 that the latter are equivalent in an appropriate sense.
- We briefly discuss the notion of (∞, n)-categories in §6 with the aim of shedding more light on the nature of $(\infty, 1)$-categories.

The main prerequisites for reading this exposition are a working knowledge of model categories, simplicial sets and enriched categories in the scope of [DS95], [Rie] and [Kel05, §§1.1-1.6] respectively. Furthermore we will make infrequent use of Bousfield localisations as well as monads and operads. An exposition of Bousfield localisations may be found in [Hir03, Ch. 3], monads are discussed in [Mac98, Ch. VI] and a precise definition of operads may be found in [May].

We will adhere to the following conventions: First of all we will disregard set theoretical issues, so for example we will not distinguish between 'small' and 'large' categories. Furthermore **Top** will refer to some convenient category of topological spaces (e.g. compactly generated spaces [McC69]; see also [Str11, §3.2] for a general discussion) and all topological spaces we consider will belong to this fixed convenient category. Finally, given any category $\mathcal{C}$, we denote by $\widehat{\mathcal{C}}$ the category of presheaves on $\mathcal{C}$, that is, all contravariant functors from $\mathcal{C}$ to **Set**.

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2. Preliminary considerations

The notion of n-categories generalises that of ordinary categories in the sense of Eilenberg and Mac Lane [EM45a] by specifying data consisting of objects, morphisms (called 1-morphisms), morphisms between its morphisms (called 2-morphisms), morphisms between its 2-morphisms and so on all the way up to n, where n is an integer or infinity. There are many examples of structures which ought to be called n-categories. Two well-known examples are the 2-category of categories **Cat** and the 2-category of topological spaces $\mathbf{Top}_2$; the objects in $\mathbf{Top}_2$ are topological spaces, 1-morphisms are continuous maps and 2-morphisms are equivalence classes of homotopies, with two homotopies being equivalent if they are linked by an endpoint preserving homotopy of homotopies [Mac98, p. 272]. These are both *strict* n-categories.

2.1. Strict n-categories.

We begin by considering two equivalent definitions of such n-categories as these illustrate two complementary approaches to higher categories. The first definition we give in full; the second is rather lengthy so we will only provide a sketch.

Definition 2.1.1. [Lei04a, Def. 1.4.1.] Let $\{\mathbf{Str}^n\mathbf{Cat}\}_{n\in\mathbb{N}}$ be the sequence of categories given inductively by

$$\mathbf{Str}^0\mathbf{Cat} := \mathbf{Set}, \quad \mathbf{Str}^{n+1}\mathbf{Cat} := (\mathbf{Str}^n\mathbf{Cat})\text{-}\mathbf{Cat}$$

(where $\mathcal{V}$-**Cat** denotes the category of $\mathcal{V}$-enriched categories). A *strict n-category* is an object of $\mathbf{Str}^n\mathbf{Cat}$ and a *functor of strict n-categories* is a morphism of $\mathbf{Str}^n\mathbf{Cat}$. ⌟

We note that for every n the data of the 1-category $\mathbf{Str}^n\mathbf{Cat}$ in fact organises into that of a strict $(n+1)$-category via internal hom. The objects of hom-strict-n-categories are n-functors, 1-morphisms are higher analogues of natural transformations, that is families of 1-morphisms in the target strict n-category indexed by the objects in the source strict n-category, 2-morphisms are certain families of 2-morphisms in the target strict n-category and so on.
For the second definition we need to introduce some auxiliary notions: First we define the *globe category*, denoted $\mathbb{G}$, as the category generated by the graph

$$\cdots \quad \underset{t^{n+1}}{\overset{s^{n+1}}{\leftleftarrows}} n \underset{t^{n}}{\overset{s^{n}}{\leftleftarrows}} n-1 \underset{t^{n-1}}{\overset{s^{n-1}}{\leftleftarrows}} \quad \cdots \quad \underset{t^{2}}{\overset{s^{2}}{\leftleftarrows}} 1 \underset{t^{1}}{\overset{s^{1}}{\leftleftarrows}} 0$$

under the equations

$$t^m \circ s^{m-1} = s^m \circ s^{m-1}, \qquad s^m \circ t^{m-1} = t^m \circ t^{m-1}$$

for all $m \geq 2$. Then for each $n \geq 0$ we denote by $\mathbb{G}_n$ the full subcategory of $\mathbb{G}$ spanned by the objects $\{0, \dots, n\}$. A presheaf on $\mathbb{G}$ is called a *globular set* and a presheaf on $\mathbb{G}_n$ an *n-globular set*. Let X be a globular or n-globular set, then we think of the elements in $X_m := X(m)$ as objects if $m = 0$ and as m-morphisms if $m \geq 1$. Given an m-morphism $f \in X_m$, if $m \geq 2$ we think of the image of f under the maps $s_m := X(s^m)$ and $t_m := X(t^m)$ as respectively its source and target $(m-1)$-morphism and if $m = 1$ as its source and target object. For $0 \leq p < m \leq n$, we write

$$X_m \times_{X_p} X_m = \left\{ (x_1, x_2) \in X_m \times X_m \;\middle|\; \begin{array}{rl} & t_{p+1} \circ \cdots \circ t_m(x_1) \\ = & s_{p+1} \circ \cdots \circ s_m(x_2) \end{array} \right\}.$$

As mentioned above, we only provide a sketch of the second definition of strict n-categories because the conditions (a)-(f) in the stated reference are fairly technical and will not be required explicitely later on.

Definition sketch 2.1.2. [Lei04a, Def. 1.4.8.] A *strict ω-category*[1] (*strict n-category*) is a globular set (n-globular set) $\mathcal{C}$ equipped with

(i) a map $\circ_p : \mathcal{C}_m \times_{\mathcal{C}_p} \mathcal{C}_m \to \mathcal{C}_m$ for each $0 \leq p < m \leq n$ called the *composition*,
(ii) a map $i_p : \mathcal{C}_p \to \mathcal{C}_{p+1}$ for each $0 \leq p < n$; the element $i_p(x)$ is written id_x and called the *identity on x*,

satisfying conditions (a) - (f) which determine sources and targets of compositions and the behaviour of units, associativity and interchange laws.
A functor of strict ω-categories (strict n-categories) is a morphism of presheaves on $\mathbb{G}$ (on $\mathbb{G}_n$) which is compatible with the identity and composition maps. The category of strict ω-categories (strict n-categories) is denoted $\mathbb{G}\mathbf{Cat}$ ($\mathbb{G}_n\mathbf{Cat}$). ⌟

Again, via internal hom the category $\mathbb{G}\mathbf{Cat}$ (the categories $\mathbb{G}_n\mathbf{Cat}$) organises into a category enriched in strict ω-categories (strict n-categories).

2.1.1. *Comparing the two definitions of strict n-categories.* The two ways of approaching higher categories referred to above are thus to either proceed inductively, defining an n-category as something built out of $(n-1)$-categories, which yields a compartmentalised organisation of the morphisms, or to specify the data all over again for each n which means considering all morphisms at once on a more equal footing.
The two definitions of strict n-categories may be seen to be equivalent by introducing a variation of the first definition, which we now describe, to which the second definition is more easily compared.
First we recall that a functor $\mathbb{G}_1^{\mathrm{op}} \to \mathcal{V}$ with composition and identity morphisms satisfying analogous conditions as in Definition sketch 2.1.2 is called an *internal category* in $\mathcal{V}$; these form a category which we denote $\mathrm{Cat}(\mathcal{V})$ (see e.g. [Mac98, §XII.1]). Thus internal categories in **Set** are just strict 1-categories according to Definition sketch 2.1.2 and these may easily be verified to be equivalent to ordinary categories. For some $\mathcal{V}$ the category $\mathrm{Cat}(\mathcal{V})$ contains a subcategory equivalent to $\mathcal{V}$-**Cat**. For example, the full subcategory of $\mathrm{Cat}(\mathbf{Top})$, spanned by objects T such that the space $T(0)$ is discrete, is equivalent to **Top**-**Cat**. Similarly, for every $n \geq 0$ the full subcategory of $\mathrm{Cat}(\mathbf{Str}^n\mathbf{Cat})$, consisting of objects C such that for $1 \leq k \leq n$ all k-morphisms in $C(0)$ are identities, is equivalent to $\mathbf{Str}^n\mathbf{Cat}$-$\mathbf{Cat}$, i.e. $\mathbf{Str}^{n+1}\mathbf{Cat}$. We now define for each $n \geq 0$ the *n-fold internal categories*; these organise into a category which we denote by $\mathbf{Cat}^n$. We set $\mathbf{Cat}^0 := \mathbf{Set}$ and then the n-fold internal categories for $n \geq 1$ are the objects C of $\mathrm{Cat}(\mathbf{Cat}^{n-1})$ such that for all $1 \leq k \leq n$ all k-morphisms in $C(0)$ are identities. The category $\mathbf{Cat}^n$ is then seen to be equivalent to $\mathbf{Str}^n\mathbf{Cat}$ by induction. Via the repeated application of internal hom in **Cat** the n-fold internal categories are equivalent to certain presheaves on the n-fold product $\mathbb{G}_1 \times \cdots \times \mathbb{G}_1$ (together with identity and composition morphisms):

$$\mathbf{Cat}(\mathbb{G}_1 \times \cdots \times \mathbb{G}_1, \mathbf{Set}) \cong \mathbf{Cat}(\mathbb{G}_1,\ \mathbf{Cat}(\mathbb{G}_1, \ldots \mathbf{Cat}(\mathbb{G}_1, \mathbf{Set}) \ldots)).$$

We now explain how to turn presheaves on $\mathbb{G}_n$ into presheaves on the n-fold product $\mathbb{G}_1 \times \cdots \times \mathbb{G}_1$. For any $k \geq 1$ there is a functor $\mathbb{G}_1 \times \mathbb{G}_k \to \mathbb{G}_{k+1}$, surjective on objects and morphisms, given by sending the objects $(1, i)$ to $i+1$ and $(0, i)$ to 0 for all $0 \leq i \leq k$. We can now put these functors together:

$$\mathbb{G}_1 \times \overset{n\times}{\cdots} \times \mathbb{G}_1 \;\to\; \mathbb{G}_1 \times \overset{n-2\times}{\cdots} \times \mathbb{G}_1 \times \mathbb{G}_2 \;\to\; \cdots \;\to\; \mathbb{G}_1 \times \mathbb{G}_{n-1} \;\to\; \mathbb{G}_n. \tag{2.1}$$

Composing the resulting functor with the underlying presheaves on $\mathbb{G}_n$ of strict n-categories yields exactly the presheaves on $\mathbb{G}_1 \times \cdots \times \mathbb{G}_1$ corresponding to n-fold internal categories.

Remark 2.1.3. If constructed with a bit of care, the equivalence between $\mathbf{Str}^n\mathbf{Cat}$ and $\mathbb{G}_n\mathbf{Cat}$ ($n \geq 0$) which we just sketched is in fact an isomorphism. Two corresponding strict n-categories under this isomorphism have bijective sets of objects and m-morphisms for all $m \in \{1, \ldots, n\}$ and also the various compositions of one strict n-category may be expressed in terms of the

[1] We write ω-category instead of ∞-category, because the latter is often used for what is more accurately called $(\infty, 1)$-category.

compositions of the other. It may then be inferred that $\mathbb{G}_n\mathbf{Cat}$ may be organised into a category enriched in $\mathbf{Str}^{n-1}\mathbf{Cat}$ (and vice versa) and then $\mathbb{G}_n\mathbf{Cat}$ and $\mathbf{Str}^n\mathbf{Cat}$ are isomorphic as $\mathbf{Str}^{n+1}\mathbf{Cat}$-categories. ⌟

2.2. **Introducing weak composition laws.** Let us return to the example of the strict 2-category of topological spaces. It seems reasonable that one should be able to define higher versions of $\mathbf{Top}_2$ by keeping track not only of homotopies between continuous maps but also also of homotopies between homotopies between continuous maps and homotopies between homotopies between homotopies between continuous maps and so on. As the convenient category $\mathbf{Top}$ is Cartesian closed the category $\mathbf{Top}_2(X, Y)$ is simply the fundamental groupoid of the mapping space $\mathcal{H}om_{\mathbf{Top}}(X, Y)$ for any pair of objects (X, Y) in $\mathbf{Top}_2$ (see [Bro06, Ch. 6] for a detailed discussion of fundamental groupoids). We may isolate the difficulty inherent in defining higher versions of $\mathbf{Top}_2$ by studying higher versions of the fundamental groupoid of topological spaces.

2.2.1. *An attempt at defining fundamental ∞-groupoids.* We begin by studying a reasonable family of objects and k-morphisms for $k \geq 1$ of the fundamental ∞-groupoid of a topological space. Intuitively, points should be objects, paths should be 1-morphisms, endpoint preserving homotopies of paths should be 2-morphisms, "end-path preserving" homotopies of 2-morphisms should be 3-morphisms and so on. We may formalise this as follows: Consider the functor $D : \mathbb{G} \to \mathbf{Top}$ given by sending every natural number i to the i-dimensional disk $D_i = \left\{ \ x \in \mathbb{R}^i \ \middle| \ \|x\| \leq 1 \ \right\}$ and by sending the morphisms s^i and t^i to the continuous maps σ^i and τ^i defined by mapping D_{i-1} to the northern and the southern hemisphere of ∂D_i respectively for all $i \geq 1$. Similarly to the total singular complex functor we may now define a functor $\mathbf{Top} \to \widehat{\mathbb{G}}$ by assigning to any topological space X the presheaf $\mathbf{Top}(D_{(\_)}, X)$. To see that for a given $i \geq 0$ we may think of $D_i \to X$ as a homotopy between homotopies and so on we proceed inductively (for simplicity we here parametrise homotopies by D_1 instead of $[0, 1]$): For every $i \geq 1$ we define the map $D_i \times D_1 \to D_{i+1}$ described by mapping $\left\{ \ (x, 1) \in D_i \times D_1 \ \middle| \ x \in D_i \ \right\}$ and $\left\{ \ (x, -1) \in D_i \times D_1 \ \middle| \ x \in D_i \ \right\}$ to the north and south pole of D_{i+1} respectively. For every $n \geq 0$ we then obtain maps

$$D_1 \times \overset{n\times}{\cdots} \times D_1 \ \to \ D_2 \times D_1 \times \overset{n-2\times}{\cdots} \times D_1 \times \ \to \ \cdots \ \to \ D_{n-1} \times D_1 \ \to \ D_n.$$

The composition of the above maps $D_1 \times \cdots \times D_1 \to D_n$ may then be composed with any given continuous map $D_n \to X$ to yield a map $D_1 \times \cdots \times D_1 \to X$. We recover the viewpoint of n-morphisms between $(n-1)$-morphisms and so on via internal hom in $\mathbf{Top}$, that is the map $D_1 \times \cdots \times D_1 \to X$ corresponds to a map $D_1 \to \mathbf{Top}(D_1, \mathbf{Top}(D_1, \ldots \mathbf{Top}(D_1, X) \ldots))$, where D_1 appears $n-1$ times in the second term.

We now discuss the composition of paths. For each $n \geq 0$ we define the *space of reparametrisations* $E(n) = \left\{ \ f \in C([0, 1], [0, n]) \ \middle| \ f(0) = 0, \ f(1) = n \ \right\}$. We fix a reparametrisation $r \in E(2)$, e.g. $t \mapsto 2t$, then given two paths $p_1, p_2 : [0, 1] \to X$ such that $p_1(1) = p_2(0)$, we denote by $p_1 \sqcup p_2 : [0, 2] \to X$ the map which sends t to $p_1(t)$ if $t \in [0, 1]$ and to $p_2(t)$ if $t \in [1, 2]$. The composition $p_2 \circ p_1$ of p_1 and p_2 is then $(p_1 \sqcup p_2) \circ r$. If we identify homotopic paths as we do in the fundamental groupoid, composition of paths and thus of homotopies between continuous maps in $\mathbf{Top}_2$ is associative. If we keep track of higher homotopies, then the two ways of composing three paths p_1, p_2, p_3, namely $p_3 \circ (p_2 \circ p_1)$ and $p_3 \circ (p_2 \circ p_1)$ are no longer equal, but the two corresponding reparametrisations r' and r'' in $E(3)$ are connected by a path $D_1 \to E(3)$, which by internal hom in $\mathbf{Top}$ and the universal property of quotient spaces corresponds uniquely to a map $f : D_2 \to [0, 3]$ such that $f \circ \sigma^2 = r'$ and $f \circ \tau^2 = r''$ (after identifying D_1 and $[0, 1]$). But any two such paths may themselves be linked by an endpoint preserving homotopy, which themselves may be linked by a higher homotopy and so on. This follows from the fact that the $E(n)$ are contractible. (Conversely any topological space with this property is weakly contractible.) In this sense composition is still essentially associative. Similarly, the composition of n paths is essentially associative. We will continue discussing the example of higher fundamental groupoids in §3.

2.2.2. *2-categories of spans.* We provide another example of a higher categorical structure with this property: Let $\mathcal{C}$ be a category with pullbacks, a *span* between two objects X_1, X_2 in $\mathcal{C}$ consists of a pair of arrows from some object S to X_1 and X_2 respectively, i.e. a diagram

$$\begin{array}{ccc} & S & \\ \swarrow & & \searrow \\ X_1 & & X_2. \end{array}$$

The object S is called the *apex* of the span. For every diagram of the shape $X \to Y \leftarrow Z$ in $\mathcal{C}$ we fix a pullback square and call it the *canonical pullback of* $X \to Y \leftarrow Z$ in $\mathcal{C}$. The *2-category of spans in* $\mathcal{C}$ has the same objects as $\mathcal{C}$, 1-morphisms are spans in $\mathcal{C}$, 2-morphisms are morphisms in $\mathcal{C}$ between apices of spans which have a common source and target making the resulting diagram commute. Composition of 1-morphisms $X_1 \leftarrow S' \to X_2$ and $X_2 \leftarrow S'' \to X_3$ is given by the canonical pullback of $S' \to X_2 \leftarrow S''$ composed with the morphisms $X_1 \leftarrow S'$ and $S'' \to X_3$. We see that composition of 1-morphisms is associative up to a canonical invertible 2-morphism. This is an example of a so-called bicategory which we discuss in §2.3. For more details as well as the similar example of the 2-category of rings and bimodules see [Mac98, §VII.7] (and for why they are similar see [Bae08]).

2.2.3. *Some properties of higher categories.* Reviewing our discussion so far, we have encountered three main properties n-categories[2] should possess according to any reasonable definition:

(a) Hom-objects of n-categories should be $(n-1)$-categories.
(b) All laws governing composition should be expressed by canonical higher invertible morphisms.
(c) The totality of all n-categories should arrange into an $(n+1)$-category.

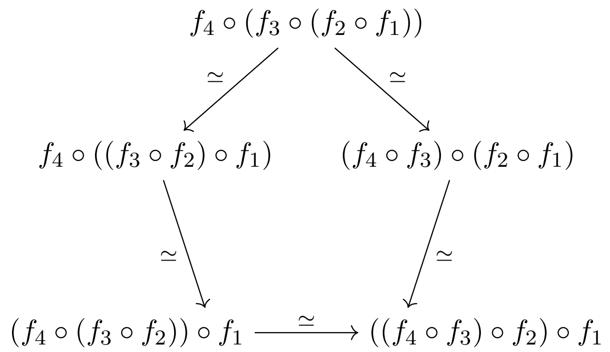
By canonical we mean that the higher invertible morphisms in question should satisfy some sort of uniqueness property. When we began defining higher fundamental groupoids the uniqueness property consisted in the fact that any choice of higher invertible morphism was itself linked by higher invertible morphisms, any two of which were in turn linked by higher invertible morphisms and so on; these higher morphisms were moreover "distinguished" in in that the corresponding homotopies factored through a homotopy in the relevant reparametrisation space. For the 2-category of spans the invertible morphisms in question were unique because they were induced by a universal property. Furthermore we would like these invertible morphisms to be compatible with the overall structure of our n-categories, e.g. the composition of two canonical invertible morphisms should again be canonical.
Such conditions characterising the canonicity and compatibility of invertible morphisms governing composition laws are loosely termed *coherence conditions.* In all definitions of higher categories we will discuss there exists some natural notion of coherence, although there seems to be no overreaching theory unifying these notions. Progress is being made however, e.g. in [JK13] laws governing units are examined in a rather general setting.

2.3. **A first attempt at defining general n-categories.** Considering the properties which we expect general n-categories should satisfy it seems reasonable to give a definition by taking either Definition 2.1.1 or Definition sketch 2.1.2 and replacing the equalities in the laws governing composition with higher invertible morphisms. The first definition of higher categories, the *bicategories* introduced by Bénabou in [Bén67] follows this approach (see also [Lei98] for a more modern treatment). As the name suggests, bicategories formalise the notion of 2-categories. Bénabou in fact gives two equivalent definitions, one in terms of objects with hom-1-categories [Bén67, Def. 1.1] and one in terms of 2-globular sets with composition maps [Bén67, §1.3] and then shows that they are equivalent [Bén67, Prop. 1.3.10]. Defining general n-categories via

[2]What we loosely call n-categories are often called *weak* n-categories. We use the term n-categories in accordance with the observation that in general examples of higher categories are of this form and should consequently not require the qualification 'weak'. When we wish to emphasise that we are considering a larger class of objects than strict n-categories we will use the term *general* n-categories.

enrichment is a priori somewhat easier because the associativity and unit rules only involving higher morphisms are already taken care of in the definition of $(n-1)$-categories and interchange laws are given by the definition of k-functors, with $1 \leq k \leq n-1$. Accordingly, the definitions to appear later on of *tricategories* [GPS95] and *tetracategories* [Tri95] follow this approach.
We will now discuss the difficulties inherent in the definition of higher categories resembling bicategories. For each iteration in the inductive definition of n-categories the main focus lies on determining composition rules of 1-morphisms and defining n-functors. Our discussion will focus on the the associativity of 1-morphisms. It will be convenient to consider any n-category as an ω-category with all morphism above n being identities.
What would reasonably be called "unicategories" are simply ordinary categories because all higher morphisms are identities. The data of a bicategory $\mathcal{C}$ should include objects, hom-categories and for any three objects X_1, X_2, X_3 a composition functor $\circ : \mathcal{C}(X_1, X_2) \times \mathcal{C}(X_2, X_3) \to \mathcal{C}(X_1, X_3)$. As announced above, we specify isomorphisms in place of the identities governing associativity in strict 2-categories. As we wish these isomorphisms governing composition to be compatible with the overall structure of bicategories we organise these into natural isomorphisms, that is for any four objects X_1, X_2, X_3, X_4 in $\mathcal{C}$ we specify a natural isomorphism between the two functors $\mathcal{C}(X_1, X_2) \times \mathcal{C}(X_2, X_3) \times \mathcal{C}(X_3, X_4) \to \mathcal{C}(X_1, X_4)$ given respectively by $(\_ \circ \_) \circ \_$ and $\_ \circ (\_ \circ \_)$. Given any sequence of composable morphisms $f_1, \ldots, f_n$ these natural isomorphisms suffice to generate isomorphisms between any two orders of composing $f_1, \ldots, f_n$. These isomorphisms are however not necessarily unique. Say we have a sequence of four composable morphisms $f_1, \ldots, f_4$, then there exist two paths from $f_4 \circ (f_3 \circ (f_2 \circ f_1))$ to $((f_4 \circ f_3) \circ f_2) \circ f_1$.

(2.2)



There should be a canonical natural 3-morphism between the two chains of 2-morphisms from $f_4 \circ (f_3 \circ (f_2 \circ f_1))$ to $((f_4 \circ f_3) \circ f_2) \circ f_1$ but 3-morphisms are identities, that is, the two chains must be required to be the same. We could now define bicategories as objects together with hom-categories, composition functors and natural isomorphisms encoding associativity satisfying the property that all 2-morphisms with the same source and target, obtained by repeatedly applying the natural isomorphism encoding associativity, should be equal by definition. It turns out that it is sufficient to require equality only for pentagons as in (2.2) for this to hold for all such pairs of 2-morphisms.
This may be deduced by examining Stasheff's *associahedra* [Sta63], [Lod04]. These constitute a sequence $\{K_n\}_{n \geq 2}$ of $(n-2)$-dimensional cell complexes homeomorphic to a ball, characterised by having k-cells in one-to-one correspondence with all meaningful ways of inserting $n-k-2$ pairs of brackets into a word of length n and such that the $(k-1)$-dimensional cells in the boundary of any k-dimensional cell are exactly those in correspondence with the bracketings obtained by inserting one extra pair of brackets. The associahedra may in fact be realised as convex polyhedra; the earliest proof we know of is [Hai81]. For example, K_3 is a closed interval and K_4 corresponds to the pentagon in (2.2). The subcells of K_n may be described as products $K_{r_1} \times \cdots \times K_{r_m}$ with $r_1 + \cdots + r_m + 1 - m = n$.
Thus, returning to bicategories, any set of chains of composable 2-morphisms, beginning and ending at the same object, obtained by applying the natural isomorphisms governing associativity in different orders may be seen as paths along edges of K_n, whose 2-dimensional cells

are either commutative squares (by naturality) or commutative pentagons (by assumption), so the compositions of these chains must coincide [SS93, §5.4]. Let us consider the next higher version, tricategories. The data of a tricategory includes objects, hom-bicategories, composition bifunctors, again denoted by $\circ$, and for every triple of objects a natural equivalence between $(\_\circ\_)\circ\_$ and $\_\circ(\_\circ\_)$, which again suffice to generate invertible 2-morphisms between all different bracketings of compositions of sequences of 1-morphisms. As 3-morphisms are no longer identities we need to define 3-morphisms between the various invertible 2-morphisms generated by associativity. By our characterisation of subcells of the associahedra, we see that if we define natural invertible 3-morphsisms from the left path in pentagons as in (2.2) to the right path, these suffice to generate 3-morphisms between all 2-morphisms generated by associativity. These have to be unique (4-morphisms are identities) and the property which ensures this is the commutativity of a certain diagram resembling K_5. Tricategories are rather complex and unwieldy; a problem which intensifies dramatically as one attempts to proceed to even higher dimensions. In [Tri95] and [Hof11, §3.1] it is argued that it should be possible for any dimension n to derive the necessary diagrams from K_{i+2} for $1 \leq i \leq n$ to give a definition of n-categories and n-functors by some formal procedure. Trimble develops these ideas to a sufficient degree in order to give a definition of tetracategories; no definition of categories of higher dimension in this sense are known. The complexity of tetracategories is staggering; the statement of the K_6 axiom alone takes up almost six pages in [Hof11, §3.2].

Despite their shortcomings, the definitions of higher categories under discussion provide useful insights about patterns we may expect to encounter in any given definition of higher categories. Bénabou already notes that monoidal categories are equivalent to bicategories with one object [Bén67, Ex. I.3.4.1.a]. Baez and Dolan expand upon this observation and make similar predictions for the larger class of *k-tuply monoidal n-categories*; these are $(n+k)$-categories which have exactly one object, one 1-morphism, one 2-morphism and so on all the way up to $k-1$ [BD95, §5]. We just saw that the notion of 1-tuply monoidal 1-categories is equivalent to that of monoidal categories; carrying on, it may be seen the notions of 2-tuply monoidal 1-categories and braided monoidal categories coincide as well [GPS95, Prop. 8.6] and it believed that this is true also for 3-tuply monoidal 1-categories and symmetric monoidal categories. Baez and Dolan conjecture that for $n+2 \leq k \leq k'$ the notions of k-tuply monoidal and k'-tuply monoidal n-categories coincide; this is the known as the *Stabilisation Hypothesis*[3].

A further fundamental observation we can make in this setting is that many categorical structures are equivalent to some stricter version in a suitable sense. Perhaps the most well known result is that every monoidal category is equivalent to a strict monoidal category and more generally that every bicategory is in fact biequivalent to a strict 2-category [Str96, Prop. 9.3]. Every tricategory is triequivalent to a Gray category, which is a certain type of tricategory which notably has strict associativity and unit laws but weak interchange laws [GPS95, Th. 8.1]. Not every tricategory is triequivalent to a strict 3-category, however. To see this, one shows that the trifunctors between tricategories with one object and one 1-morphism, hence braided monoidal categories, are exactly the functors of braided monoidal categories. Because the notions of strict braided and strict symmetric monoidal categories coincide and because any braided monoidal category which is equivalent to a symmetric monoidal category as a braided monoidal category is itself symmetric, one sees that assuming that every tricategory is triequivalent to a strict 3-category would imply that all braided monoidal categories are symmetric [GPS95, Rem. 8.8]. This provides our first example that the notion of n-categories is strictly more general that that of strict n-categories.

Faced with the apparent inadequacy of higher categories in the sense of bicategories a plethora of other definitions have been proposed (for detailed overviews we recommend [Lei02], [Lei04a, Ch. 10] and [CL04]). There is as of yet no satisfactory description of how they relate to each other, and in particular, in which sense they may be equivalent.

[3] Versions of the stabilisation hypothesis have been proved, notably [Lur12, Ex. 5.1.2.3].

2.4. **Defining general higher categories using operads or monads.** In order to circumvent the combinatorial complexity of composing chains of morphisms one may replace binary composition with composition parametrised by operads. Trimble [Tri99] and May [May01] have independently suggested such an approach, specifically, May proposes a general definition of operadic categories of which Trimble's higher categories form particular examples. May first specifies a ground category $\mathcal{B}$, equipped with a symmetric monoidal structure and a compatible model structure as well as an A_∞-operad $\mathcal{P}$ in $\mathcal{B}$, that is for every $k \in \mathbb{N}$ the object $\mathcal{P}(k)$ is weakly contractible. Then given a symmetric monoidal $\mathcal{B}$-enriched model category $(\mathcal{G}, \otimes)$ and a bifunctor $\odot : \mathcal{B} \times \mathcal{G} \to \mathcal{G}$ satisfying certain conditions, he defines a $\mathcal{P}$-*category in* $\mathcal{G}$ *over a set* O as a family of objects in $\mathcal{G}$ indexed by pairs in O (denoted $\mathcal{G}(X, Y)$ for $X, Y \in O$) and for every sequence $X_0, \ldots, X_k$ of objects in O a composition morphism

$$\mathcal{P}(k) \odot \mathcal{G}(X_0, X_1) \otimes \cdots \otimes \mathcal{G}(X_{k-1}, X_k) \to \mathcal{G}(X_0, X_k),$$

such that all composition morphisms together are compatible with the operad structure.
To define n-categories one chooses a suitable ground category, again denoted by $\mathcal{B}$, together with an A_∞-operad $\mathcal{P}$ in $\mathcal{B}$ and then proceeds inductively: One begins by specifying the base case of 0-categories. Having defined n-categories, one arranges these into a symmetric monoidal model category $n\mathbf{Cat}$ and defines a bifunctor $\odot : \mathcal{B} \times n\mathbf{Cat} \to n\mathbf{Cat}$. An $(n+1)$-category is then a $\mathcal{P}$-category in $n\mathbf{Cat}$. In Trimble's definition the ground category $\mathcal{B}$ is **Top** (with the standard model structure and the Cartesian monoidal structure), the objects constituting the operad $\mathcal{P}$ are the spaces $\{E(k)\}_{k\in\mathbb{N}}$ (see §2.2.1) and for each $n \in \mathbb{N}$ the bifunctor $\odot$ is given by composing a fundamental n-groupoid functor $\mathbf{Top} \to n\mathbf{Cat}$ with the product in $n\mathbf{Cat}$.
May's approach also encompasses the theory of A_∞-categories, which were introduced in [Fuk93]. The category $\mathcal{B}$ is again **Top**, the objects constituting the operad $\mathcal{P}$ are the associahedra $\{K_n\}_{n\in\mathbb{N}}$ (see §2.3; $K_1 = K_0 = \{*\}$), the category $\mathcal{G}$ is the category of chain complexes over a field k (with the standard model structure and tensor product), denoted $\mathbf{Ch}_k$, and the bifunctor $\odot$ is given by composing the cellular chain complex functor with the tensor product in $\mathbf{Ch}_k$ [MSS02, §§1.8,1.9], [LV12, §9.2].
Operads and their cousins monads may also be used to define general n-categories resembling the definition of strict n-categories as n-globular sets. There is a free-forgetful adjunction between $\mathbb{G}_n\mathbf{Cat}$ and the category of reflexive n-graphs; these are n-globular sets X with morphisms $i_p : X_p \to X_{p+1}$ such that $s_{p+1} \circ i_p = t_{p+1} \circ i_p = \mathrm{id}_{X_p}$. Strict n-categories may then be defined as algebras of reflexive n-graphs over the monad induced by this adjunction. Penon [Pen99] weakens this definition by defining n-categories as algebras over a certain monad on reflexive n-graphs and Batanin [Bat98] and Leinster [Lei04b] as algebras over a certain operad on reflexive n-graphs. Again, coherence is expressed in terms of contractibility.

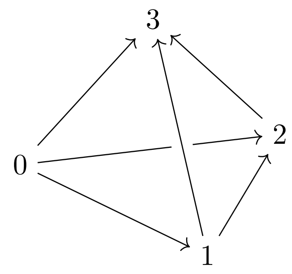
In view of these last two approaches to higher categories and the discussion of higher groupoids in §2.2.1 it seems reasonable to consider the following alternative to property (b) any reasonable definition of n-categories should satisfy.

(b′) All laws governing composition should be expressed by the contractibility of the spaces of all possible choices of composition or the contractibility of spaces parametrising such choices.

2.5. **A geometric approach.** A strategy which is now especially popular is to abandon the notion of explicit composition altogether. This is the approach we will focus on in this exposition. Basically, we think of higher categories, not as consisting of objects and morphisms together with *composition maps* obeying certain rules, but as consisting of objects and morphisms which must *fit together* in a certain way; consequently we will speak of the *geometric* approach to higher categories[4]. We begin by discussing such an approach to (strict) 1-categories.

[4]A glance at [CL04, Appendix A] reveals that there are many characteristics according to which one may group various definitions of higher categories, e.g. Leinster [Lei04a, Ch. 10] distinguishes between algebraic and

2.5.1. *The nerve construction.* Denote by Δ the simplex category, then the fact that any partially ordered set may be interpreted as a category and that order preserving maps are precisely the functors between such categories determines a canonical embedding $\Delta \hookrightarrow \mathbf{Cat}$. For each $n \in \mathbb{N}$ we denote the category corresponding to $[n]$ by Δ_n and the objects of Δ_n by $\{0, \ldots, n\}$. The arrows of Δ_n excluding the identities form the directed edges of the standard n-simplex. Consider for example Δ_3:



We may now define a functor $N : \mathbf{Cat} \to \mathbf{SSet}$ by assigning to every category $\mathcal{C}$ the simplicial set $\mathbf{Cat}(\Delta_{(\_)}, \mathcal{C})$, called the *nerve* of $\mathcal{C}$. In [FGA, Prop. III.4.1.] it is first noted that this functor is fully faithful and a slight variant of the condition discussed bellow is given to characterise the simplicial sets isomorphic to nerves of categories. The condition is named after Graeme Segal who further studied the relationship between categories and simplicial sets (notably in [Seg68]).

2.5.2. *The Segal condition.* Let X be a simplicial set. For $n \geq 1$ and $1 \leq k \leq n$ we define the maps

$$\alpha_k := X \left(\begin{array}{c} [1] \to [n] \\ 0 \mapsto (k-1), \;\; 1 \mapsto k \end{array} \right). \tag{2.3}$$

Now we arrange n copies of X_1 into the following diagram

$$X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} \cdots \xleftarrow{d_0} X_1 \xrightarrow{d_1} X_0 \xleftarrow{d_0} X_1 \tag{2.4}$$

on which X_n forms the vertex of a cone by adding the maps $\alpha_1, \ldots \alpha_n$ from X_n to the X_1s in the same order as the X_1s are arranged in (2.4). This induces a canonical map

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

called the *Segal map*, where the object on the right denotes the limit over (2.4). A simplicial set is said to satisfy the *Segal condition* if for all $n \geq 1$ the Segal map is a bijection.

2.5.3. *Categories as simplicial sets.* We explain how to view any simplicial set X satisfying the Segal condition as a category. This will serve as a basic example to help us understand the definitions presented later on. The central idea, as is to be expected, is to view X as the nerve of some category $\mathcal{C}$. The set X_0 may then be identified with the objects of $\mathcal{C}$ and X_1 with its morphisms where $d_0, d_1 : X_1 \to X_0$ assign sources and targets respectively. To recover composition we begin by fixing the concrete construction of the limits $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ given by $\left\{ \; (f_1, \ldots, f_n) \in X_1 \times \cdots \times X_1 \;\middle|\; d_1(f_i) = d_0(f_{i+1}), \; 0 \leq i \leq n-1 \; \right\}$, so its elements are sequences of morphisms with matching sources and targets. The preimage of a sequence of morphisms $(f_1, \ldots, f_n)$ under the canonical bijection $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is then to be thought of as the unique simplex corresponding to the free category generated by these morphisms; we call this n-simplex the *composition of* $f_1, \ldots, f_n$. Applying $X([1] \to [n], \; 0 \mapsto 0, 1 \mapsto n)$ then yields the edge $f_n \circ \cdots \circ f_1$ which we call the *result* of the composition[5]. There is a canonical faithful functor $\mathbb{G}_1 \hookrightarrow \Delta$ sending $i \mapsto i, \; i = 0, 1$ and thus $X_1 \rightrightarrows X_0$ is a 1-globular set. The identities satisfied by the face and degeneracy maps of Δ together with the Segal condition ensure that $s_0 : X_0 \to X_1$ sends every object to its

non-algebraic definitions and Schommer-Pries [SP12] refers to different perspectives emphasising enrichment, presheaves and monads.

[5]The distinction between composition and result will be crucial later on.

corresponding identity map, that the inverse of $X_2 \to X_1 \times_{X_0} X_1$ composed with d_1 is the composition map in Definition sketch 2.1.2 and that these satisfy the appropriate axioms.
The main point here is that we have repackaged the information determining internal categories in **Set** into the source category of the relevant presheaves and we are then able to encode composition in terms of a simple property.

2.6. (∞, n)**-categories.** We have now discussed several approaches to handling the complexity inherent in the coherence conditions of higher categories. Here we turn to the question of enrichment and discuss Lurie's motivation for the introduction of (∞, n)-categories [Lur09a, p. 6]; these are ω-categories such that all k-morphisms are invertible for $k > n$. He notes that a naïve approach to a definition of n-categories via enrichment would be circular; when defining what an n-category is we make reference to the totality of all $(n-1)$-categories, but these are supposed to already form an n-category (see §2.2.3). In the strict case we get around this by first forming the strict 1-category of strict $(n-1)$-categories from which we may reconstruct the strict n-category of strict $(n-1)$-categories via internal hom. As the composition of functors between general $(n-1)$-categories is a priori not associative this approach does not carry over directly. Lurie notes that we do not need all higher morphisms to formulate rules such as associativity; we only need higher invertible morphisms. Thus if we have already defined $(\infty, n-1)$-categories, we may arrange them into the underlying $(\infty, 1)$-category of their (not yet defined) (∞, n)-category over which we may enrich.

2.7. **Plan for our discussion of higher categories using a geometric approach.** The point of departure for our development of higher categories will be that we wish to find a geometric definition of (∞, n)-categories via enrichment (although as mentioned in the abstract our main focus will be on $(\infty, 1)$-categories). Our discussion will occasionally require significant backtracking; to forestall the disorientation this may cause we now provide a brief plan of what is to follow.
We begin our plan by letting the cat out of the bag: Model categories will turn out to be particularly pleasant incarnations of $(\infty, 1)$-categories; we will always arrange the totality of all (∞, n)-categories and (∞, n)-functors according to any given definition into a subcategory of a model category and consider this model category the underlying $(\infty, 1)$-category of the $(\infty, n+1)$-category of (∞, n)-categories. We will often call such a model category a *model of (∞, n)-categories*. The fibrant objects will generally correspond to (∞, n)-categories[6] and the remaining objects to "pre-(∞, n)-categories", objects which possess some but not all of the properties of (∞, n)-categories; we will often refer to the data which constitutes the structure of pre-(∞, n)-categories as objects, morphism, composition etc. in so far as this makes sense. Weak equivalences in these model categories will correspond to equivalences of (∞, n)-categories.
Recall that the inductive definition of strict n-categories presupposes a definition of strict 1-categories and we have to explicitly specify the base case, that is, strict 0-categories. The situation is the same for (∞, n)-categories, so our discussion will proceed in three steps:

1. We give a definition of $(\infty, 0)$-categories in §3.
2. In §4 we present five different definitions of $(\infty, 1)$-categories and we show in §5 that there are chains of Quillen equivalences linking all corresponding model categories.
3. In §6 we may finally proceed inductively. We will in fact find that we have learnt enough about higher categories to understand not only definitions based on enrichment but also ones resembling strict n-categories as n-globular sets. Furthermore the definitions of higher (∞, n)-categories will shed some light on the definitions of $(\infty, 1)$-categories.

3. $(\infty, 0)$-CATEGORIES

In §2.2.1 we saw that any topological space should give rise to an ∞-groupoid. Let x_0 be a point in X, then for all $i \geq 1$ we may interpret the homotopy group $\pi_i(X, x_0)$ as the path

[6] In most models of (∞, n)-categories considered here the cofibrations are monomorphisms and thus all objects are cofibrant.

components of the $(i-1)$-tuply ∞-groupoid whose sole object is x_0 (see §2.3). Recall that for a non-negative n and a group G (Abelian if $n > 1$), any path-connected space $K(G, n)$ such that $\pi_n(X) \cong G$ and $\pi_q(X) \cong 0$ for $q \neq n$ is called an *Eilenberg-Mac Lane space* of the pair (G, n). Such spaces were first considered in [EM45b] and their existence shown in [Whi49b]; any such space is unique up to weak equivalence, in fact, the fundamental group functor induces an equivalence of categories

$$\pi_1 : \mathrm{Ho}(\{\text{Pointed Eilenberg-Mac Lane spaces } K(G,n), \text{ with } n=1\}) \xrightarrow{\simeq} \mathbf{Grp} \tag{3.1}$$

and for integers $i > 1$ the homotopy group functors induce equivalences of categories

$$\pi_i : \mathrm{Ho}(\{\text{Pointed Eilenberg-Mac Lane spaces } K(G,n), \text{ with } n=i\}) \xrightarrow{\simeq} \mathbf{Ab} \tag{3.2}$$

(this may be derived for example from [Str11, Th. 17.44]). We thus see that at least simple spaces are determined up to homotopy by their fundamental ∞-groupoids; indeed, it is easily seen that the functor in (3.1) extends to the equivalence

$$\Pi_1 : \mathrm{Ho}(\{\text{Spaces } X \text{ with } \pi_i(X,x) = 0 \text{ for all } x \in X,\ i > 1\}) \xrightarrow{\simeq} \mathrm{Ho}(\mathbf{Str}\text{-}1\text{-}\mathbf{Grpd}). \tag{3.3}$$

Whitehead introduced the notion of *crossed module* in [Whi49a]; these are purely algebraic objects given by two groups M and P, an action of P on M and a homomorphism $M \to P$, satisfying certain properties; they come with a notion of weak equivalence. In [MW50] it is shown that there is an equivalence of categories

$$\mathrm{Ho}(\{\text{Connected, pointed spaces } X \text{ with } \pi_i(X) = 0,\ \ i > 2\}) \xrightarrow{\simeq} \mathrm{Ho}(\{\text{Crossed modules}\}) \tag{3.4}$$

(for details and a modern proof see [Bau91, § III.8.2]). Ronald Brown and his collaborators at Bangor generalised the notion of crossed modules to that of crossed complexes in order to model even more homotopy classes of spaces [BH81]; Brown communicated his results to Grothendieck who thereupon sent his famous letter to Daniel Quillen which evolved into "Pursuing Stacks" [Gro83]. There he discusses ideas for definitions of n-groupoids ($n \leq \infty$), such that a fundamental groupoid functor analogous to the one in (3.3) should yield an equivalence of categories:

$$\mathrm{Ho}(\{\text{Spaces } X \text{ with } \pi_i(X) = 0,\ \ i > n\}) \xrightarrow{\simeq} \mathrm{Ho}(n\text{-}\mathbf{Grpd}) \tag{3.5}$$

The assertion of the existence of a satisfactory notion of n-groupoids and a fundamental n-groupoid functor inducing an equivalence between homotopy categories as in (3.5) is often referred to as the *homotopy hypothesis*.

Analogously to how the functor π_1 in (3.1) may be extended to the functor Π_1 in (3.3), the functor in (3.4) may be extended to a fundamental 2-groupoid functor inducing an equivalence between the homotopy category of topological spaces X with $\pi_i(X, x) = 0$ for all $x \in X,\ i > 2$ and the homotopy category of *strict* 2-groupoids [MS93]; however, this is no longer true for strict n-groupoids with $n \geq 3$ (as we might expect should happen for some $n \geq 1$ from our discussion in §2.2.1). Simpson provides a definition encoding the minimal expectations for a realisation functor [Sim12, § 2.4], that is a functor which constructs topological spaces out of n-groupoids and which induces an inverse to the fundamental n-groupoid functor on the level of homotopy categories; he then extends a result of Berger [Ber99, Rmk. 3.7] that no realisation of any strict 3-groupoid is homotopy equivalent to the 2-sphere[7]. Stephen Lack has shown that Gray-groupoids (see §2.3) model all spaces X with $\pi_i(X, x) = 0$ for all $x \in X,\ i > 3$ [Lac11] (for a similar result see [JK07]). This yields a second, more concrete example than the one at the end of §2.3 illustrating that the notion of strict higher categories is unsatisfactory.

In 2006 Maltsiniotis noticed that the ideas presented in [Gro83] were in fact sufficiently worked out to provide a rigorous definition of ∞-groupoids, which may be generalised to a notion of ω-category very similar to that of Batanin and Leinster, which we touched upon in §2.4 [Mal07].

[7] In fact, the notion of n-groupoids considered by Simpson is a slightly weakened version introduced by Kapranov and Voevodsky in [KV91], where they claim they prove the homotopy hypothesis; their realisation functor is one in the sense of Simpson, so their result appears to contain an error.

Maltsiniotis and Ara have developed the theory of Grothendieck's ∞-groupoids to the point where we have a well understood notion of weak equivalence between such ∞-groupoids and a fundamental ∞-groupoid functor inducing a functor between homotopy categories. Their fundamental ∞-groupoids contain the same objects and morphisms as the one we started discussing in §2.2.1 and the composition of 1-morphisms is also essentially the same. It still remains to be proved that the functor between homotopy categories induced by the fundamental ∞-groupoid functor is indeed an equivalence of categories [Ara13].

3.1. **A model of $(\infty, 0)$-categories.** In view of the close connection between various types of groupoids and spaces, the homotopy hypothesis is often taken as a definition. Lurie writes [Lur09a, p. 5]: "The converse, which asserts that every $(\infty, 0)$-category has the form $\pi_{\leq n}X$ for some topological space X, is a generally accepted principle of higher category theory." 'Space' is often taken to mean 'Kan complex', the fibrant objects in **SSet** (with the Quillen model structure). This is justified for a number of reasons: The total singular complex of any topological space is a Kan complex and as this functor is part of a Quillen equivalence it induces an equivalence of categories between the homotopy category of topological spaces and the homotopy category of Kan complexes. The fundamental 1-groupoid functor factors through the category of Kan complexes via the total singular complex functor; its inverse as well as the inverse of the fundamental 2-groupoid functor are constructed by composing the nerve of 1-groupoids or 2-groupoids, which map to Kan complexes, with the geometric realisation functor. Thus we can think of Kan complexes as being intermediaries between spaces and ∞-groupoids. This view is further supported by the observation that Kan complexes behave much like nerves of categories. Indeed, it may be inferred from our characterisation of Kan complexes in §4.3 that the Segal maps of any Kan complex K are surjective and thus for any sequence of composable morphisms $(f_1, \ldots, f_n) \in K_1 \times_{K_0} \cdots \times_{K_0} K_1$ there exists a not necessarily unique element k in the preimage of $K_n \to K_1 \times_{K_0} \cdots \times_{K_0} K_1$ which we again call the composition of $(f_1, \ldots, f_n)$. Applying $K([1] \to [n],\ 0 \mapsto 0, 1 \mapsto n)$ to k yields an element in K_1 which we call its result. In this situation the distinction between compositions and results is more pronounced than for nerves of categories (discussed in §2.5.1). If K is the total singular complex associated to a topological space X, then for a given sequence of composable paths both the set of compositions as well as the set of results of compositions are subsets of mapping spaces and thus inherit a topology. The space of compositions is always contractible. This is not necessarily true for the space of results of compositions (although being the image of a contractible and hence path-connected space under a continuous map it is path-connected). Consider for example the 2-sphere together with the pair of composable paths given by first going from the north pole to the equator and then from the equator to the south pole.

Definition 3.1.1. An $(\infty, 0)$-*category* is a Kan complex and a *functor of* $(\infty, 0)$-*categories* is a morphism of simplicial sets. ⌟

As announced in our plan in §2.7, the $(\infty, 0)$-categories constitute the fibrant objects in **SSet** equipped with the Quillen model structure [DS95, § 11.1].

3.2. **Path components of simplicial sets.** We will make repeated use of the notion of path components of a simplicial set so we recall some key definitions and facts here. Let X be a simplicial set. Given two points $x, y \in X$ we say that x *is strictly homotopic to* y if there exists an edge $c \in X_1$ such that $d_0(c) = x$ and $d_1(c) = y$. We write $x \to y$ if x is strictly homotopic to y. Any point in X is strictly homotopic to itself. We say that x and y are *homotopic* if they are equivalent with respect to the equivalence relation generated by the relation of being strictly homotopic. The elements x and y are thus homotopic iff there exists a path connecting the two, that is there exist points $z_1, \ldots, z_n$ such that

$$x \longrightarrow z_1 \longleftarrow z_2 \longrightarrow \ \cdots \ \longrightarrow z_{n-1} \longleftarrow z_n \longrightarrow y.$$

The equivalence classes on X_0 created by this equivalence relation are called the *path components of* X_0. The functor taking simplicial sets to their set of path components is denoted by π_0. This functor commutes with finite products (see e.g. [JT08, Prop. 3.2.1]). The functor π_0 maps Quillen weak equivalences in **SSet** to bijections.

4. $(\infty, 1)$-categories

Our main reference for this section and the next is [Ber10], which presents four different models of $(\infty, 1)$-categories: *simplicial categories, complete Segal spaces, Segal categories* and *quasi-categories*. Barwick and Kan have shown that in addition *relative categories* provide an equivalent model of $(\infty, 1)$-categories [BK12b]. We will discuss simplicial categories and relative categories in §4.1, Segal spaces, in particular complete Segal spaces and Segal categories, in §4.2 and Quasi-categories in §4.3. Unfortunately the standard terminology employed for the corresponding model categories does not reflect the special status of fibrant objects mentioned in §2.7, e.g. the fibrant simplicial categories have no special name.
We note that most notions discussed here where initially conceived and studied in the context of homotopy theory.

4.1. Simplicial categories and relative categories.

4.1.1. *Simplicial categories.* According to our plan in §2.7 the data constituting an $(\infty, 1)$-category should include objects and hom-$(\infty, 0)$-categories. Surprisingly, the simple-minded approach using (strict) enrichment works:

Definition 4.1.1. A *simplicial category* is a category enriched in simplicial sets. ⌟

There is a risk of confusion since it would make more sense to use the term 'simplicial category' for a simplicial object in **Cat**; a more precise term would be 'simplicially enriched category' (which is in fact also sometimes used). The two notions may however be precisely related: The category of simplicial categories is canonically isomorphic to the subcategory of simplicial objects in **Cat** such that the face and degeneracy functors are constant on objects.
This approach should not a priori give a suitable definition of $(\infty, 1)$-categories; after all, composition of 1-morphisms shouldn't be strict in general. This seems however to agree with the observation made in §2.3, that certain stricter versions of a general definition of a given notion of higher category will yield an equivalent structure, e.g. we mentioned that every bicategory is biequivalent to a strict 2-category.
Let (X, Y) be a pair of objects in a simplicial category $\mathcal{C}$. The vertices of $\mathcal{C}(X, Y)$ are called 1-*morphisms.* Two 1-morphisms $f_1, f_2 \in \mathcal{C}(X, Y)$ are called *homotopic* if they lie in the same path component of $\mathcal{C}(X, Y)$ (discussed in §3.2). We write $f_1 \sim f_2$ if f_1 and f_2 are homotopic. A 1-morphism $f \in \mathcal{C}(X, Y)$ is called a *homotopy equivalence* if there exists a 1-morphism $g \in \mathcal{C}(Y, X)$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.
We now describe the right notion of equivalence between simplicial categories. Given a simplicial category $\mathcal{C}$, we denote by $\pi_0\mathcal{C}$ the *component category*, which has the same objects as $\mathcal{C}$ and hom-set $\pi_0\mathcal{C}(X, Y)$, the path components of $\mathcal{C}(X, Y)$, for any pair objects (X, Y). This makes sense as the functor $\pi_0 : \mathbf{SSet} \to \mathbf{Set}$ commutes with finite products. Clearly a 1-morphism is a homotopy equivalence iff it gets mapped to an isomorphism by π_0. A *Dwyer-Kan equivalence* or simply *DK-equivalence* is a simplicial functor $F : \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:

(a) For any pair of objects (X, Y) in $\mathcal{C}$, the map of simplicial sets

$$\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$$

is a weak equivalence.

(b) The induced functor on component categories $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

Remark 4.1.2. It is straightforward to check that an equivalent definition of DK-equivalence is obtained by replacing condition (b) with the condition

(b′) The induced functor on component categories $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is essentially surjective.

Thus DK-equivalences may be viewed as simplicial functors satisfying properties which are analogous to those for functors of ordinary categories of being fully faithful and essentially surjective. ⌟

Bergner proves the existence of a suitable model structure on **SSet-Cat** [Ber07c, 1.1]:

Theorem 4.1.3. *There is a model category structure on the category of simplicial categories defined by the following three classes of morphisms:*

(1) *The weak equivalences are the DK-equivalences.*
(2) *The fibrations are the maps $F : \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:*
   - *For any pair of objects (X, Y) in $\mathcal{C}$, the map*
   $$\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$$
   is a fibration of simplicial sets.
   - *For any objects X_1 in $\mathcal{C}$, Y in $\mathcal{D}$, and homotopy equivalence $e : F(X_1) \to Y$ in $\mathcal{D}$, there is an object X_2 in $\mathcal{C}$ and homotopy equivalence $d : X_1 \to X_2$ in $\mathcal{C}$ such that $F(d) = e$.*
(3) *The cofibrations are the maps which have the left lifting property with respect to the acyclic fibrations.* □

We will denote by $\mathbf{Cat}_\Delta$ the category **SSet-Cat** equipped with this model structure.
The fibrant simplicial categories are exactly the categories enriched in Kan complexes [Lur09a, Th. A.3.2.24]; a characterisation of cofibrant simplicial categories may be found in [Ber07c, Def. 2.2].
We finish this subsection by verifying that ordinary categories may be viewed as simplicial categories. The restriction of the total simplicial complex functor to the category of discrete topological spaces induces a fully faithful functor $N_d : \mathbf{Set} \hookrightarrow \mathbf{SSet}$. We then obtain a fully faithful functor $\mathbf{Cat} \hookrightarrow \mathbf{SSet\text{-}Cat}$ by sending a category $\mathcal{C}$ to the simplicial category with the same objects as $\mathcal{C}$ and hom-simplicial-set $N_d(\mathcal{C}(X, Y))$ for all pairs of objects (X, Y) in $\mathcal{C}$. It is easily verified that N_d has a left adjoint given by the path component functor π_0 and a right adjoint given by the functor which sends every simplicial set X to X_0. These adjunctions may be used to show that the functor $\mathbf{Cat} \hookrightarrow \mathbf{SSet\text{-}Cat}$ has a left adjoint given by the component category functor π_0 and a right adjoint which assigns to any simplicial category $\mathcal{C}$ the category $\mathcal{C}_0$ with the same objects as $\mathcal{C}$ and hom-sets $\mathcal{C}(X, Y)_0$ for all pairs of objects (X, Y) in $\mathcal{C}$. We call this category the *underlying category of* $\mathcal{C}$.

4.1.2. *Relative categories.* Quillen introduced the notion of model category as a general setting to do homotopy theory and the Quillen equivalence to provide a criterion for when two model categories should be considered equivalent [Qui67]. He writes however [Qui67, p. I.04-I.05]: "The definition of homotopy theory[8] associated to a model category is obviously unsatisfactory. [. . .] Presumably there is higher order structure [. . .] on the homotopy category which forms part of the homotopy theory of a model category, but we have not been able to find an inclusive general definition of this structure with the property that this structure is preserved when there are adjoint functors which establish an equivalence of homotopy theories.". Furthermore Quillen appears to consider *simplicial* model categories the more natural setting to do homotopy theory; these are simplicial categories $\mathcal{M}$ which underlying category $\mathcal{M}_0$ (defined in §4.1.1) is equipped with a model structure which is compatible with the simplicial structure of $\mathcal{M}$. The homotopy relation of 1-morphisms described in §4.1.1 is generally finer than that of left and right homotopy, but they agree for 1-morphisms from cofibrant objects to fibrant ones (see [Hir03, Ch. 9] for a modern treatment of simplicial model categories). Quillen writes [Qui67, p. II.0.2]: "However there are certain categories of differential graded algebras which do not seem to have natural simplicial structures but which are model categories, which is the main reason for the generality in Chapter I [on non-simplicial model categories]." Topological spaces,

[8]Here 'homotopy theory' means the homotopy category of a model category, together with extra structure if the model category is pointed.

simplicial sets and simplicial groups are shown to organise into simplicial model categories in [Qui67, Ch. II]. In a series of papers [DK80c], [DK80a], [DK80b] Dwyer and Kan address these points. They call attention to the fact that the homotopy category of a model category only depends on the weak equivalences and make the notion precise that a category together with a distinguished subcategory is actually all that is needed to provide an appropriate setting to do homotopy theory.

Definition 4.1.4. A *relative category* is a pair of categories $(\mathcal{C}, \mathcal{W})$, where $\mathcal{W}$ is a subcategory of $\mathcal{C}$, containing all identities, called the *weak equivalences of* $\mathcal{C}$. A *relative functor* between relative categories $F : (\mathcal{C}_1, \mathcal{W}_1) \to (\mathcal{C}_2, \mathcal{W}_2)$ is a functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ which restricts to a functor from $\mathcal{W}_1$ to $\mathcal{W}_2$. The category of relative categories is denoted **RelCat**. ⌟

Thus any model category together with its subcategory of weak equivalences is a relative category. Let us consider a relative category $(\mathcal{C}, \mathcal{W})$. The key insight of Dwyer and Kan is that taking the localisation $\mathrm{Ho}(\mathcal{C})$ kills too much information; they argue that one should consider its richer *simplicial localisation* $L(\mathcal{C}, \mathcal{W})$ [DK80c, §4.1]. This is a simplicial category with the same objects as $\mathcal{C}$ and which has the property that the component category (defined in §4.1.1) is canonically isomorphic to $\mathrm{Ho}(\mathcal{C}, \mathcal{W})$. If $\mathcal{C}$ is a simplicial model category, then for every cofibrant object X and every fibrant object Y in $\mathcal{C}$ the simplicial sets $L(\mathcal{C}_0, \mathcal{W}_0)(X, Y)$ and $\mathcal{C}(X, Y)$ are linked by a finite zigzag of weak equivalences. Simplicial localisation is functorial, in particular it takes relative functors to simplicial functors. A Quillen equivalence in general does not consist of relative functors, but the restriction of the right adjoint to fibrant objects and dually the restriction of the left adjoint to cofibrant objects are relative functors (see [DS95, Lemma 9.9]) which induce a pair of DK-equivalences[9]. The induced functors between the component categories recovers the equivalence of categories given by deriving the Quillen equivalence.
The relative functors which induce DK-equivalences form the weak equivalences of a model structure on **RelCat** [BK12a] and are themselves called *DK-equivalences*; we will briefly discuss this model structure in §5.1.1 and see that relative categories provide a model of $(\infty, 1)$-categories. The fact that relative categories are considered to contain the minimal amount of information needed to determine homotopy theories justifies the slogan: "The study of $(\infty, 1)$-categories is the study of homotopy theories.", in fact, as we are studying the underlying $(\infty, 1)$-category of the $(\infty, 2)$-category of $(\infty, 1)$-categories we are studying the "homotopy theory of homotopy theories"; similarly §6 could be entitled "The homotopy theory of higher categories".
We finish here by noting that any category may be seen as a relative category by either considering its subcategory consisting of all its identities or its underlying groupoid, that is the subcategory consisting of all isomorphisms, as its weak equivalences.

4.2. **Segal spaces.** We noted in §4.1.1 that simplicial categories may be thought of as an equivalent stricter version of $(\infty, 1)$-categories. Here we present a definition of $(\infty, 1)$-categories, *Segal spaces*, without such strictness properties. The name derives from Segal's Γ-spaces[10] [Seg74]. In §2.5.1 we gave an equivalent definition of ordinary categories as certain simplicial sets. As sets are to ordinary categories what Kan complexes are to $(\infty, 1)$-categories it seems plausible that $(\infty, 1)$-categories should be able to be defined as certain simplicial objects in **SSet**. We denote the category of *simplicial simplicial sets*, that is functors $\Delta^{\mathrm{op}} \to \mathbf{SSet}$ by **SSSet**. Via internal hom in **Cat** we may view any simplicial simplicial set as a functor $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathbf{Set}$; such a functor is called a *bisimplicial set*. We will make frequent use of this identification and usually speak of bisimplicial sets.
As in the case of simplicial sets we would like to characterise those bisimplicial sets W for which it makes sense to think of $W_{0\bullet}$ as a space of objects and $W_{n\bullet}$ as a space of n-tuples of composable morphisms for $n > 1$. Analogous morphisms α_k as those for simplicial sets in (2.3)

[9] In [DHKS04] the relationship between Quillen functors and relative functors between the underlying relative categories of model categories is studied in detail.

[10] Both Γ-spaces and operads, upon which we have already touched, were introduced to formalise the notion of algebraic structures with composition rules only given up to coherent homotopy [May72].

may be defined for simplicial objects in any category, so we again have canonical morphisms

$$W_{n\bullet} \to W_{1\bullet} \times_{W_{0\bullet}} \cdots \times_{W_{0\bullet}} W_{1\bullet}$$

for all $n \geq 1$ which are similarly called *Segal morphisms*. Like before, we say that a bisimplicial set satisfies the Segal condition if the Segal morphisms are all isomorphisms. This condition is too strong for our needs, but if we consider the category of simplicial sets with the Quillen model structure we may weaken the Segal condition by only requiring that the Segal maps be weak equivalences; we call this the *weak Segal condition*[11].

The weak Segal condition alone is not enough to ensure that bisimplicial sets behave sufficiently much like categories; we do however obtain a good notion of $(\infty, 1)$-categories if we restrict to fibrant bisimplicial sets in the Reedy model structure. On bisimplicial sets the Reedy model structure coincides with the injective model structure, that is, weak equivalences and cofibrations are levelwise weak equivalences and cofibrations of simplicial sets; fibrations are determined by the right lifting property with respect to acyclic cofibrations (for a discussion of the Reedy model structure on bisimplicial sets see [GJ99, IV 3.]; for a general discussion see [Hir03, Ch.15]). The following definition is due to Charles Rezk [Rez01].

Definition 4.2.1. A Reedy fibrant bisimplicial set satisfying the weak Segal condition is called a *Segal space.* ⌟

4.2.1. *Segal spaces as* $(\infty, 1)$*-categories.* Let $\mathcal{C}$ be a Segal space. Points in $\mathcal{C}_{0\bullet}$ are called *objects* and points in $\mathcal{C}_{1\bullet}$ are called 1*-morphisms* of $\mathcal{C}$. Because $\mathcal{C}$ is Reedy fibrant its Segal morphisms are acyclic fibrations and are thus surjective [Rez01, §4.1]. As in the discussions in §2.5.3 and §3.1 any pair of morphisms (f, g) of $\mathcal{C}$ such that $d_1(f) = d_0(g)$ may thus be *composed* by choosing any point k in the preimage of (f, g) under the Segal map $\mathcal{C}_{2\bullet} \to \mathcal{C}_{1\bullet} \times_{\mathcal{C}_{0\bullet}} \mathcal{C}_{1\bullet}$. The *result* of this composition is defined as $d_1(k)$ which we denote by $g \circ f$. Thus, unlike for simplicial sets satisfying the Segal condition but similar to Kan complexes, composition is only defined up to a choice. This choice is canonical in the following sense: Because the Segal map $\mathcal{C}_{2\bullet} \to \mathcal{C}_{1\bullet} \times_{\mathcal{C}_{0\bullet}} \mathcal{C}_{1\bullet}$ is an acyclic fibration the *space of compositions* $\mathrm{Comp}_{f,g}$ of f and g, defined by the pullback diagram

$$\begin{array}{ccc} \mathrm{Comp}_{f,g} & \longrightarrow & \mathcal{C}_{2\bullet} \\ \downarrow & & \downarrow \\ \{(f,g)\} & \hookrightarrow & \mathcal{C}_{1\bullet} \times_{\mathcal{C}_{0\bullet}} \mathcal{C}_{1\bullet} \end{array}$$

is contractible. The space of results of compositions is therefore connected. Similarly, for any sequence $(f_1, \ldots, f_n)$, $n \geq 1$, such that for all $i \in \{1, \ldots, n-1\} : \ d_1(f_i) = d_0(f_{i+1})$, we may define the space of compositions $\mathrm{Comp}_{f_1,\ldots,f_n}$, which is again contractible.

For any pair of objects (X, Y) in $\mathcal{C}$ the mapping space $\mathcal{C}(X, Y)$ is defined by the pullback diagram

$$\begin{array}{ccc} \mathcal{C}(X,Y) & \longrightarrow & \mathcal{C}_{1\bullet} \\ \downarrow & & \downarrow{\scriptstyle d_0 \times d_1} \\ \{(X,Y)\} & \hookrightarrow & \mathcal{C}_{0\bullet} \times \mathcal{C}_{0\bullet} \,. \end{array}$$

Because $\mathcal{C}$ is assumed to be Reedy fibrant the map $d_0 \times d_1$ is a fibration, so $\mathcal{C}(X, Y)$ is a Kan complex. Similarly we may for any finite sequence of objects $(X_1, \ldots, X_n)$ define the mapping space $\mathcal{C}(X_1, \ldots, X_n)$ and the relevant Segal morphism restricts to an acyclic fibration

$$\mathcal{C}(X_1, \ldots, X_n) \xrightarrow{\sim} \mathcal{C}(X_1, X_2) \times \cdots \times \mathcal{C}(X_{n-1}, X_n).$$

For any object X in $\mathcal{C}$ we have $d_0 \circ s_0(X) = d_1 \circ s_0(X)$ so $s_0(X)$ lies in $\mathcal{C}(X, X)$. We write $\mathrm{id}_X := s_0(X)$ and call id_X the *identity* 1*-morphism of* X.

[11] This is often simply called the 'Segal condition'.

4.2.2. *Homotopy of* 1*-morphisms.* Two 1-morphisms $f, g \in \mathcal{C}(X, Y)$ are called *homotopic* if they lie in the same path component of $\mathcal{C}(X, Y)$ (see §3.2). We write $f \sim g$ if f and g are homotopic. The homotopy relation reflects the canonicity expressed by the contractibility of spaces of compositions [Rez01, Prop. 5.4]:

1. For a triple of composable 1-morphisms f, g, h we have $h \circ (g \circ f) \sim (h \circ g) \circ f$, which may be shown by finding $k', k'' \in \mathrm{Comp}_{f,g,h}$ such that $h \circ (g \circ f) = d_1 \circ d_1(k')$ and $d_1 \circ d_2(k'') = (h \circ g) \circ f$.
2. For any morphism $f \in \mathcal{C}(X, Y)$ we have $\mathrm{id}_Y \circ f \sim f \sim f \circ \mathrm{id}_X$, which follows from the observation that $s_0(f) \in \mathrm{Comp}_{f,\mathrm{id}_Y}$ and $d_1 \circ s_0(f) = f$ and dually $s_1(f) \in \mathrm{Comp}_{\mathrm{id}_X,f}$ and $d_1 \circ s_1(f) = f$.

Furthermore, a 1-morphism $f \in \mathcal{C}(X, Y)$ is called a *homotopy equivalence* if there exist $g_1, g_2 \in \mathcal{C}(Y, X)$ such that $f \circ g_1 \sim \mathrm{id}_Y$ and $g_2 \circ f \sim \mathrm{id}_X$. By properties 1. and 2. we have $g_1 \sim g_2 \circ f \circ g_1 \sim g_2$. Identity morphisms are homotopy equivalences and any morphism homotopic to a homotopy equivalence is itself a homotopy equivalence.
Homotopy between 1-morphisms is compatible with composition, for if we have $f_1, f_2 \in \mathcal{C}(X, Y)$ such that $f_1 \sim f_2$ and $g \in \mathcal{C}(Y, Z)$, then (f_1, g) and (f_2, g) lie in the same path component of $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$ and because the map $\mathcal{C}(X, Y, Z) \to \mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$ is an acyclic fibration, the path components of $\mathcal{C}(X, Y, Z)$ are in bijection to those of $\mathcal{C}(X, Y) \times \mathcal{C}(Y, Z)$, thus all choices of composition for (f_1, g) and (f_2, g) are in the same path component and thus we have $g \circ f_1 \sim g \circ f_2$.
Like for simplicial categories we may thus for any Segal space $\mathcal{C}$ define its *component category* $\pi_0\mathcal{C}$ which has the same objects as $\mathcal{C}$ and hom-set $\pi_0\mathcal{C}(X, Y)$ for any pair of objects (X, Y).

4.2.3. *The space of objects* $\mathcal{C}_{0\bullet}$. Let $\mathcal{C}$ be a Segal space. Denote by $\mathcal{C}_{\mathrm{hoequiv}} \subseteq \mathcal{C}_{1\bullet}$ the space consisting exactly of those components which points are homotopy equivalences. Because the morphism of simplicial sets $s_0 : \mathcal{C}_{0\bullet} \to \mathcal{C}_{1\bullet}$ has a left inverse it is a monomorphism (specifically a levelwise injective map) and it takes objects to homotopy equivalences, that is the map s_0 factors through $\mathcal{C}_{\mathrm{hoequiv}}$. Let (X, Y) be a pair of objects in $\mathcal{C}$, then we define the *space of paths from x to y* denoted $\mathrm{Path}_{X,Y}$ by the pullback diagram

$$\begin{array}{ccc} \mathrm{Path}_{X,Y} & \longrightarrow & \mathcal{C}_{0\bullet} \\ \downarrow & & \downarrow \\ \{(X,Y)\} & \hookrightarrow & \mathcal{C}_{0\bullet} \times \mathcal{C}_{0\bullet}. \end{array}$$

By the universal property of pullbacks there exists a canonical morphism (which is easily seen to be a monomorphism) from $\mathrm{Path}_{X,Y}$ to the subspace of $\mathcal{C}(X, Y)$ consisting of homotopy equivalences (see diagram (6.3) in [Rez01]). We may thus view $\mathcal{C}_{0\bullet}$ as a sub ∞-groupoid of the ∞-groupoid obtained by throwing away all the non-invertible 1-morphisms of $\mathcal{C}$.

4.2.4. *The model category of Segal spaces.* The left Bousfield localisation of **SSSet** together with the Reedy model structure in which the Segal spaces are exactly the fibrant objects exists by [Rez01, Th. 7.1]. This model structure is called the *Segal space model category structure* and we denote the category of bisimplicial sets with this model structure by **SegSp**. By construction a Reedy weak equivalence between any two objects in **SegSp** is a weak equivalence in the Segal space model category structure, and if both objects are themselves Segal spaces then the converse holds. The cofibrations are the monomorphisms and thus all objects are cofibrant.
Unfortunately, the model category **SegSp** does not have the right class of weak equivalences.

A morphism $F : \mathcal{C} \to \mathcal{D}$ between two Segal spaces is again called a *DK-equivalence* if it satisfies the following two conditions:

(a) For any pair of objects (X, Y) in $\mathcal{C}$, the map of simplicial sets
$$\mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$$
is a weak equivalence.

(b) The induced functor on component categories $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ is an equivalence of categories.

As for DK-equivalences between simplicial categories, an equivalent notion of DK-equivalence between Segal spaces is obtained by merely requiring $\pi_0 F : \pi_0\mathcal{C} \to \pi_0\mathcal{D}$ to be essentially surjective.

Proposition 4.2.2. *Any weak equivalence between Segal spaces in the Reedy model structure is a DK-equivalence.*

Proof. This follows from [Rez01, Th. 7.7] and the observation that Reedy weak equivalences are weak equivalences in the complete Segal space model structure considered in §4.2.5. □

The converse does not hold in general; DK-equivalences do not register how "saturated" the space of objects is with homotopy equivalences, which of course Reedy weak equivalences do, being levelwise weak equivalences of simplicial sets.

4.2.5. *Complete Segal spaces.* We may form the left Bousfield localisation of **SegSp** with respect to the DK-equivalences to obtain a new a new model category denoted by **CSS** called the *complete Segal space model structure.*

Proposition 4.2.3. *The fibrant objects of* **CSS** *are exactly the Segal spaces* $\mathcal{C}$ *such that* $s_0 : \mathcal{C}_{0\bullet} \hookrightarrow \mathcal{C}_{hoequiv}$ *is a weak equivalence of simplicial sets.*

Proof. Combine [Rez01, Th. 7.7] with the observation that the morphism $F(0) \to E$ used to define the model structure in [Rez01, §12] is a DK-equivalence. □

Definition 4.2.4. A *complete Segal space* is a Segal space $\mathcal{C}$ such that the inclusion $s_0 : \mathcal{C}_{0\bullet} \hookrightarrow \mathcal{C}_{\mathrm{hoequiv}}$ is a weak equivalence of simplicial sets. ⌟

By [Rez01, Prop. 6.2] complete Segal spaces are the Segal spaces $\mathcal{C}$ such that for all pairs of objects (X, Y) in $\mathcal{C}$ the canonical map from $\mathrm{Path}_{X,Y}$ to the space of homotopy equivalences from X to Y is a weak equivalence. We may thus view complete Segal spaces as those Segal spaces $\mathcal{C}$ for which $\mathcal{C}_{0\bullet}$ is equivalent to the $(\infty, 0)$-category obtained by throwing away all the non-invertible 1-morphisms of $\mathcal{C}$.
The fact that not all Segal spaces are fibrant in **CSS** is not a problem. Denote by $\mathbf{SegSp}^f$ and $\mathbf{CSS}^f$ the full subcategories of **SSSet** spanned by Segal spaces and complete Segal spaces respectively. If we view these categories are relative categories with weak equivalences being DK-equivalences then the canonical inclusions $\mathrm{Ho}(\mathbf{CSS}^f) \hookrightarrow \mathrm{Ho}(\mathbf{SegSp}^f) \hookrightarrow \mathrm{Ho}(\mathbf{CSS})$ are both equivalences of categories by construction. In fact, the complete Segal spaces form a particularly pleasant class of Segal spaces, as a morphism between complete Segal spaces is a DK-equivalence iff it is a Reedy weak equivalence (again by construction). This is useful because Reedy weak equivalences are easier to identify than DK-equivalences. The cofibrations are again the monomorphisms.

4.2.6. *Segal categories.* We now consider a notion which predates that of Segal spaces.

Definition 4.2.5. A *Segal category* is a bisimplicial set W which satisfies the weak Segal condition and for which $W_{0\bullet}$ is discrete. ⌟

Segal categories first appeared in the literature as 'special diagrams of simplicial sets' in [DKS89]. The name 'Segal category' seems to be due to Simpson [Sim97].
Reedy fibrant Segal categories are thus Segal spaces which are defined by a property dual to

that of complete Segal spaces in that their object spaces contain no homotopy equivalences. Reedy fibrant Segal categories provide a setting in which it becomes strikingly clear that Reedy equivalences are not the right notion of equivalences between Segal spaces; a Reedy equivalence between Reedy fibrant Segal categories restricts to a *bijection* between the sets of objects and is thus too strong. It may be inferred from Theorem 5.1.5 that, like for complete Segal spaces, the canonical inclusion of the homotopy category of Reedy fibrant Segal categories in the homotopy category of Segal spaces (with weak equivalences being DK-equivalences) is an equivalence of categories.

We define the *nerve functor* N : **SSet-Cat** $\to$ **SSet** which takes a simplicial category $\mathcal{C}$ to the bisimplicial set defined by $N(\mathcal{C})_{0\bullet} := \mathrm{Obj}(\mathcal{C})$ and for all $n \geq 1$

$$N(\mathcal{C})_{n\bullet} := \coprod_{(X_0,\ldots,X_n)\in\mathrm{Obj}(\mathcal{C})^{n+1}} \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{n-1}, X_n)$$

together with the obvious face and degeneracy maps. Similarly as for categories, those bisimplicial sets arising as the nerve of simplicial categories correspond exactly to the bisimplicial sets W which satisfy the Segal condition and for which $W_{0\bullet}$ is discrete; we may thus view the notion of Segal categories as a weakening of that of simplicial categories.

Remark 4.2.6. The category **SSet-Cat** may itself be endowed with the structure of a simplicial category such that for any pair of simplicial categories $(\mathcal{C}, \mathcal{D})$ the vertices of the mapping space $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ are the simplicial functors from $\mathcal{C}$ to $\mathcal{D}$[12]. The functor N : **SSet-Cat** $\to$ **SSet** may now seem more deserving of the title 'nerve functor' when we note that the nerve of any simplicial category $\mathcal{C}$ is given by $N(\mathcal{C}) : [n] \mapsto \mathrm{Hom}(\Delta^n, \mathcal{C})$, where we view Δ^n as a simplicial category via the canonical embedding **Cat** $\hookrightarrow$ **SSet-Cat** for all $n \geq 0$ (see [Joy07, §5.3]). ⌟

We now turn to the homotopy theory of Segal categories. A *Segal precategory* is a bisimplicial set W such that $W_{0\bullet}$ is discrete. We denote the full subcategory of **SSet** spanned by Segal precategories by **SegCat**. Bergner defines two model structures, both containing the same weak equivalences, on **SegCat** [Ber07d, § 5, § 7]. One reason the underlying category of the model category containing the Reedy fibrant Segal categories is chosen to be **SegCat** instead of **SSet** is because it is in general difficult to map into discrete simplicial sets. This would pose a serious obstruction to taking fibrant replacements if Reedy fibrant Segal categories are to be the fibrant objects [MO29728].

We first discuss the *injective model structure*. Unlike for the complete Segal space model structure we cannot obtain a model structure on **SegCat** by localising a model structure with weak equivalences and cofibrations being levelwise weak equivalences and cofibrations respectively because no model structure with these weak equivalences exists on **SegCat** [Ber07d, §3.12]. It is clear that the weak equivalences between Reedy fibrant Segal categories should be DK-equivalences. To define weak equivalences on all of **SegCat** Bergner proceeds as follows: The factorisation axiom of the model structure on **SegSp** may be chosen to be functorial and we then obtain a fibrant replacement functor on **SegSp**. Bergner adapts this functor to a functor L_c : **SegCat** $\to$ **SegCat** so that it takes Segal precategories to Segal spaces with discrete spaces of objects. A map of Segal precategories is then defined to be a *weak equivalence* if it is mapped to a DK-equivalence under L_c. The injective model structure is given in the following theorem:

Theorem 4.2.7. [Ber07d, Th. 5.1] *There is a cofibrantly generated model category structure on the category of Segal precategories with*

(1) *weak equivalences as defined above,*
(2) *cofibrations being the monomorphisms (in particular, every Segal precategory is cofibrant) and*
(3) *fibrations being the maps with the right lifting property with respect to the acyclic cofibrations.*

□

[12]We are unaware whether this simplicial structure is compatible with the model structure of $\mathbf{Cat}_\Delta$.

The category of Segal precategories with this model structure is denoted $\mathbf{SegCat}_c$. Bergner verifies that this model category has the correct fibrant objects:

Theorem 4.2.8. [Ber07b] *The fibrant objects in* $\mathbf{SegCat}_c$ *are exactly the Reedy fibrant Segal categories, that is Segal spaces with discrete spaces of objects.* □

The second model category structure on **SegCat** is called the *projective model category structure* and is denoted $\mathbf{SegCat}_f$. Reedy fibrant Segal categories are also fibrant in $\mathbf{SegCat}_f$ but the converse is false (this follows from Theorem 5.1.1, Theorem 5.1.4 and the discussion at the end of §5.2). Its main use is to establish a Quillen equivalence between Segal precategories and simplicial categories and we will not discuss it any further.

4.2.7. *The nerve functor* $N : \mathbf{RelCat} \to \mathbf{SSSet}$. Let $(\mathcal{C}, \mathcal{W})$ be a relative category and $\mathcal{I}$ a category. Given two functors $F, G : \mathcal{I} \to \mathcal{C}$, a natural transformation $\eta : F \Rightarrow G$ is called a *weak equivalence* if for all X in $\mathcal{I}$ the morphism η_X is in $\mathcal{W}$. We write $\mathbf{We}(\mathcal{I}, \mathcal{C})$ for the category consisting of all functors from $\mathcal{I}$ to $\mathcal{C}$ and weak equivalences between them. For $(\mathcal{C}, \mathcal{W})$ we may now define its nerve $N(\mathcal{C}, \mathcal{W})$ as the bisimplicial set given by

$$N(\mathcal{C}, \mathcal{W})_m = N(\mathbf{We}([m], \mathcal{C}))$$

where the N on the right-hand side of the equation denotes the nerve of ordinary categories (see §2.5.1). Thus, if we view the category $[m] \times [n]$ as an m-by-n grid of objects with rows of m composable horizontal arrows and columns of n composable vertical arrows, then the set of n-simplices of $N(\mathcal{C}, \mathcal{W})_m$ corresponds to the set of functors $[m] \times [n] \to \mathcal{C}$ in which the vertical arrows are sent into $\mathcal{W}$. While the nerve of a relative category satisfies the Segal condition it is in general not Reedy fibrant and thus not necessarily a Segal space [Ber09, p. 538]. If the relative category in question is the underlying relative category of a model category, then its nerve is Reedy weakly equivalent to a complete Segal space [BK13].

We now apply the nerve functor to ordinary categories. If we denote by $\mathrm{id}(\mathcal{C})$ the subcategory of $\mathcal{C}$ consisting of all identity morphisms in $\mathcal{C}$, we may consider the *discrete nerve* $N(\mathcal{C}, \mathrm{id}(\mathcal{C}))$. The discrete nerve construction embeds **Cat** as a full subcategory of **SSSet**. This embedding is the same functor as the one obtained by composing the embedding $\mathbf{Cat} \hookrightarrow \mathbf{Cat}_\Delta$ with the nerve $N : \mathbf{Cat}_\Delta \hookrightarrow \mathbf{SSSet}$ (defined in §4.2.6). Discrete nerves of categories are Reedy fibrant Segal categories [Rez01, Ex. 4.4].

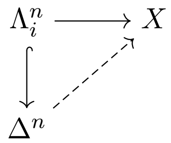
Let $\mathrm{iso}(\mathcal{C})$ denote the groupoid consisting of all isomorphisms in $\mathcal{C}$, then $N(\mathcal{C}, \mathrm{iso}(\mathcal{C}))$ is a complete Segal space called the *classifying diagram of* $\mathcal{C}$ [Rez01, Prop. 6.1]. The functor sending any category to its classifying diagram is fully faithful and a functor between categories is an equivalence iff it is sent to a DK-equivalence [Rez01, Th. 3.7].

By [Rez01, Lemma 14.2] there is a DK-equivalence $N(\mathcal{C}, \mathrm{id}(\mathcal{C})) \to N(\mathcal{C}, \mathrm{iso}(\mathcal{C}))$.

Remark 4.2.9. In [Lur09c, §2.1] Lurie gives a vivid description of how to view complete Segal spaces as "classifying diagrams of $(\infty, 1)$-categories". ⌟

4.3. **Quasi-categories.** Quasi-categories were introduced by Boardman and Vogt who from the outset noted their similarity to categories [BV73, p. 102][13]. They in a sense provide the most economical definition of $(\infty, 1)$-categories as they are just certain simplicial sets and we will see that they offer the most direct approach to defining weak composition. The most exhaustive published reference on quasi-categories is [Lur09a]; a further detailed (but unpublished) exposition may be found in [Joy08b]; the short exposition [Rie08] is a great primer.

Recall that for a simplicial set X we say that a horn $\Lambda^n_i \to X$ has a *filler* if it can be extended as in the following diagram:



[13] In [BV73] they are simply called simplicial sets satisfying the restricted Kan condition.

Kan complexes K may be characterised as a those simplicial sets for which every horn $\Lambda^n_i \to K$ has a filler. The total simplicial complex of any topological space is a Kan complex; this follows from the fact that $|\Lambda^n_i|$ is a retract of the simplex $|\Delta^n|$. In §2.5.1 we characterised nerves of categories as simplicial sets satisfying the Segal condition. We now provide a characterisation in terms of filling horns.

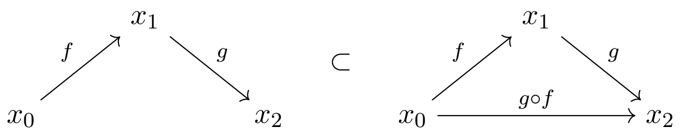
Proposition 4.3.1. [Lur09a, Prop. 1.1.2.2.] *Let X be a simplicial set. Then the following conditions are equivalent:*

(1) *There exists a category $\mathcal{C}$ and an isomorphism $X \cong N(\mathcal{C})$.*
(2) *For each $n \geq 2$ and each $0 < i < n$ every horn $\Lambda^n_i \to X$ has a* unique *filler.*

$\square$

The horns Λ^n_i such that $0 < i < n$ are called *inner horns*. The proof of Proposition 4.3.1 is a bit fiddly but it is easy to see why it should be true. Consider a category $\mathcal{C}$ and an inner horn $\Lambda^n_2 \to N(\mathcal{C})$. Then this horn has a unique filler, which may be represented by the following picture:

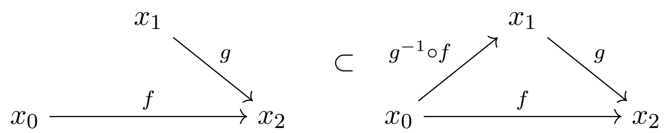
(4.1)



There is an evident way of weakening this condition; we simply require inner horns to have *some* filler. Then the face on the right of (4.1) constitutes some composition between $x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2$ and $x_0 \xrightarrow{g \circ f} x_2$ its result as in our discussions in §§2.5.3, 3, 4.2.1.

Definition 4.3.2. A *quasi-category* is a simplicial set $\mathcal{C}$ such that for all $n \geq 2$ and for all $0 < i < n$ any map $\Lambda^n_i \to \mathcal{C}$ admits an extension to $\Delta^n \to \mathcal{C}$. ⌟

In particular Kan complexes are quasi-categories. But for Kan complexes also the outer horns have fillers. For a category $\mathcal{C}$ filling a horn of the type $\Lambda^2_0 \to N(\mathcal{C})$ would correspond to the following picture:



The categories with the property indicated by this diagram and its counterpart for Λ^2_2 are exactly groupoids, so it makes sense to think of Kan complexes as those quasi-categories which are ∞-groupoids.
Let $\mathcal{C}$ be a quasi-category, the we call its points *the objects of* $\mathcal{C}$ and its vertices the 1*-morphisms of* $\mathcal{C}$. For any object X of $\mathcal{C}$ the 1-morphism $s(X)$ is called the *identity morphism of* X, which we denote by id_X. The composition and the result of a composition are as discussed above.
We will not discuss the notion of coherence for quasi-categories beyond noting that associativity is again in a sense governed by Stasheff's associahedra (discussed in §2.3). For a detailed discussion of associativity we refer the reader to [Rie09].
Two 1-morphisms $f, g : X \to Y$ are called *homotopic* if equivalently there exists either a 2-simplex which boundary is made up of f, g, id_X or one which boundary is made up of f, g, id_Y. We write $f \sim g$ if f and g are homotopic. A 1-morphism $f : X \to Y$ is called a *homotopy equivalence* if there exists a morphism $g : Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$. The homotopy relation is an equivalence relation [Lur09a, Prop. 1.2.3.5] and we may again form the *homotopy category of* $\mathcal{C}$ which has the same objects as $\mathcal{C}$ and which morphisms are homotopy classes of 1-morphisms of $\mathcal{C}$ [Lur09a, Prop. 1.2.3.7].
The description of mapping spaces of quasi-categories is less explicit than in the other definitions

of $(\infty,1)$-categories, however, they continue to become better understood; in [DS11] even a notion of DK-equivalence between simplicial sets is introduced which is equivalent to the notion of weak equivalence which we discuss in §4.3.1.

4.3.1. *The model category of quasi-categories.* We finish by giving a brief description of the model category which contains the quasi-categories. The underlying category is the category of simplicial sets. We will need the fact that **SSet** is Cartesian closed (for a proof see e.g. [MM92, Prop. I.6.1]). First we note that a map of simplicial sets $X \to Y$ is a Quillen weak equivalence iff for every Kan complex K the map $\pi_0(\mathcal{H}om_{\mathbf{SSet}}(Y,K)) \to \pi_0(\mathcal{H}om_{\mathbf{SSet}}(X,K))$ is a bijection of sets [Qui67, Prop. 2.3.4]. The weak equivalences in the model structure of quasi-categories are defined similarly: Denote by τ_1 the left adjoint to the nerve functor $N : \mathbf{Cat} \to \mathbf{SSet}$. We can then define a functor $\tau_0 : \mathbf{SSet} \to \mathbf{Set}$ which takes a simplicial set X to the set $\tau_0(X)$ of isomorphism classes of objects of the category $\tau_1(X)$, then for any pair (X,Y) of simplicial sets we define $\tau_0(X,Y) := \tau_0(\mathcal{H}om_{\mathbf{SSet}}(X,Y))$. A *weak categorical equivalence* is a map $A \to B$ of simplicial sets such that the induced map $\tau_0(B,X) \to \tau_0(A,X)$ is a bijection for every quasi-category X. Every weak categorical equivalence is a Quillen weak equivalence [Joy08b, Prop. 6.15]. Joyal proves [Joy08b, Th. 6.12]:

Theorem 4.3.3. *There is a model category structure on the category of simplicial sets with the following properties:*

(1) *Weak equivalences are the weak categorical equivalences.*
(2) *The cofibrations are the monomorphisms (in particular, every simplicial set is cofibrant).*
(3) *The fibrant objects are the quasi-categories.*

□

We denote the category of simplicial sets with this model structure by **QCat**.

5. Comparing different definitions of $(\infty,1)$-categories

In this section we first show that there are sufficiently many Quillen equivalences to establish that all models of $(\infty,1)$-categories determine the same homotopy theory and then we summarise how the various definitions of $(\infty,1)$-categories qualitatively relate to each other.

5.1. **Quillen equivalences between models of $(\infty,1)$-categories.** Recall first that the composition of two Quillen equivalences is again a Quillen equivalence, provided that one composes only left and right adjoint functors, and secondly that any of the two constituent functors of a Quillen equivalence determines the other up to unique isomorphism. For all Quillen equivalences described bellow only one out of its two constituent functors will be given explicitly.
From $\mathbf{Cat}_\Delta$ there are three functors which establish Quillen equivalences, one to $\mathbf{SegCat}_f$, one to $\mathbf{SegCat}_c$ and one to **QCat**.
The first is the nerve functor $N : \mathbf{Cat}_\Delta \to \mathbf{SegCat}_f$ defined in §4.2.6. We denote its left adjoint by F.

Theorem 5.1.1. [Ber07d, 8.6] *The adjoint pair*

$$F : \mathbf{SegCat}_f \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{Cat}_\Delta : N$$

is a Quillen equivalence. □

Next we study the *homotopy coherent (or simplicial) nerve* $\widetilde{N} : \mathbf{Cat}_\Delta \to \mathbf{QCat}$, introduced by Cordier and Porter in [CP86] which constitutes the right adjoint of a Quillen equivalence. Like any right adjoint functor with source **SSet** it may be represented by a cosimplicial object in its target category, in this case by $\mathfrak{C}\Delta : \Delta \to \mathbf{Cat}_\Delta$, which we now describe following [Rie11]. As we discussed in §2.4 the forgetful functor from $\mathbb{G}_n\mathbf{Cat}$ to reflexive n-graphs admits a left adjoint. For $n = 1$ reflexive n-graphs are just reflexive graphs, i.e. directed graphs equipped with a distinguished edge at each vertex called the identity edge. The adjunction between categories and reflexive graphs determines a comonad on **Cat**. Resolving any category by this comonad

produces a simplicial object in **Cat** with face and degeneracy maps constant on objects; it may thus be viewed as a simplicial category by the embedding of simplicial categories in the category of simplicial objects in **Cat** mentioned in §4.1.1. If we view categories as simplicial categories, then this comonadic resolution takes a category to a cofibrant replacement in $\mathbf{Cat}_\Delta$. For every $n \geq 0$ the object $\mathfrak{C}\Delta^n$ is the comonadic resolution of Δ^n. Thus $\mathfrak{C}\Delta^n$ has the same objects as Δ^n and mapping spaces freely generated by the morphisms in Δ^n.
We denote by J the left adjoint of the homotopy coherent nerve. The following theorem appears to have been proved independently by Lurie [Lur06, §1.3.4] and Joyal [Joy07, Th. 2.10].

Theorem 5.1.2. *The adjoint pair*

$$J : \mathbf{QCat} \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Cat}_\Delta : \widetilde{N}$$

is a Quillen equivalence. □

We now describe the last of the three Quillen functors connecting $\mathbf{Cat}_\Delta$ to the other models of $(\infty, 1)$-categories, the *strong coherent nerve* $K^! : \mathbf{Cat}_\Delta \to \mathbf{SegCat}_c$. We noted in Remark 4.2.6 that the category of simplicial categories may itself be equipped with the structure of a simplicial category. The strong coherent nerve of a simplicial category $\mathcal{C}$ is then defined as $K^!(\mathcal{C}) : [n] \mapsto \mathrm{Hom}(\mathfrak{C}\Delta^n, \mathcal{C})$. We denote its left adjoint by $K_!$.

Theorem 5.1.3. [Joy07, Cor. 2.9] *The adjoint pair*

$$K_! : \mathbf{SegCat}_c \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{Cat}_\Delta : K^!$$

is a Quillen equivalence. □

The Quillen equivalence between the two model structures for Segal categories is easy.

Theorem 5.1.4. [Ber07d, Th. 7.5] *The adjoint pair*

$$\mathrm{id} : \mathbf{SegCat}_f \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{SegCat}_c : \mathrm{id}$$

is a Quillen equivalence. □

Since the underlying category of **CSS** is the category of bisimplicial sets and the underlying category of $\mathbf{SegCat}_c$ is the category of bisimplicial sets discrete in degree zero, there is an inclusion functor $I : \mathbf{SegCat}_c \to \mathbf{CSS}$. This functor has a right adjoint $R : \mathbf{CSS} \to \mathbf{SegCat}_c$ which may be thought of as a "discretisation functor". This functor takes complete Segal spaces to Reedy fibrant Segal categories and the counit of this adjunction evaluated at a complete Segal space $\mathcal{C}$ yields a DK-equivalence $R(\mathcal{C}) \to \mathcal{C}$.

Theorem 5.1.5. [Ber07d, Th. 6.3] *The adjoint pair*

$$I : \mathbf{SegCat}_c \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{CSS} : R$$

is a Quillen equivalence. □

Joyal and Tierney provide two Quillen equivalences between both **CSS** and **QCat** as well as between $\mathbf{SegCat}_c$ and **QCat** [JT07]. In both cases the two Quillen equivalences have the opposite orientation. All but one of the specified functors with source **QCat** are obtained by discarding data.
We begin by considering the functor $i_1^* : \mathbf{SSSet} \to \mathbf{SSet}$ which assigns to each bisimplicial set W the simplicial set $W_{\bullet 0}$. If W is a Segal space, then $i_1^*(W)$ is a quasi-category [JT07, Cor. 3.6]. We denote the left adjoint of i_1^* by p_1^*.

Theorem 5.1.6. *The adjoint pair*

$$p_1^* : \mathbf{QCat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{CSS} : i_1^*$$

is a Quillen equivalence. □

We denote the restriction of i_1^* to **SegCat** by j^* and its left adjoint by q^*.

Theorem 5.1.7. *The adjoint pair*

$$q^* : \mathbf{QCat} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{SegCat}_c : j^*$$

is a Quillen equivalence. □

Remark 5.1.8. The homotopy coherent nerve functor $\widetilde{N}$ is equal to the composition of the strong coherent nerve functor $K^!$ and the the functor j^*. ⌟

Instead of extracting the first row we may extract the diagonal to form a Quillen equivalence. Concretely, denote by $d^* : \mathbf{SegCat}_c \to \mathbf{QCat}$ the functor which sends a Segal precategory W to the simplicial set $[n] \mapsto W_{n\,n}$ and its right adjoint by d_*.

Theorem 5.1.9. *The adjoint pair*

$$d^* : \mathbf{SegCat}_c \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{QCat} : d_*$$

is a Quillen equivalence. □

The second Quillen equivalence between **CSS** and **QCat** is given by a total simplicial set functor $t_! : \mathbf{CSS} \to \mathbf{QCat}$ and its right adjoint $t^!$.

Theorem 5.1.10. *The adjoint pair*

$$t_! : \mathbf{CSS} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathbf{QCat} : t^!$$

is a Quillen equivalence. □

There is as of yet no known direct Quillen equivalence between simplicial categories and complete Segal spaces, although further research has gone into studying functors from simplicial categories to complete Segal spaces [Ber09].

5.1.1. *Relative categories.* We now briefly discuss the model structure on **RelCat** and the functors relating **RelCat** to other model categories.
To establish the model structure Barwick and Kan first construct a functor $N_\xi : \mathbf{RelCat} \to \mathbf{CSS}$ which is Reedy equivalent to the nerve functor N discussed in §4.2.7 [BK12b]. They use the functor N_ξ to lift the model structure on **CSS** to one on **RelCat**. Thus **CSS** and **RelCat** are Quillen equivalent by construction. In [BK12a] they prove that the weak equivalences in this category are exactly the DK-equivalences.
There are several variants of the standard simplicial localisation functor introduced in [DK80c, §4.1] for example the hammock localisation functor [DK80a, §2.1]. As different such relative functors are more convenient in different situations Barwick and Kan define in general a *simplicial localisation functor* to be any relative functor naturally DK-equivalent to the standard simplicial localisation functor [BK12a]. There is however a preferred choice of relative functor going in the other direction, the *relativisation functor* $R : \mathbf{Cat}_\Delta \to \mathbf{RelCat}$. We now need the following definitions:

Definition 5.1.11. Two relative functors $F', F'' : (\mathcal{C}_1, \mathcal{W}_1) \to (\mathcal{C}_2, \mathcal{W}_2)$ are called *homotopic* if they may be linked by a finite zigzag of weak equivalences (see §4.2.7). A relative functor $F : (\mathcal{C}_1, \mathcal{W}_1) \to (\mathcal{C}_2, \mathcal{W}_2)$ is called a *homotopy equivalence* if there exists a relative functor $G : (\mathcal{C}_2, \mathcal{W}_2) \to (\mathcal{C}_1, \mathcal{W}_1)$ such that $G \circ F$ and $F \circ G$ are homotopic to the respective identity functors. The relative functor G is called the *homotopy inverse* of F. ⌟

The relativisation functor as well as any simplicial localisation functor is a homotopy equivalence and it is shown that a relative functor $\mathbf{RelCat} \to \mathbf{Cat}_\Delta$ is a simplicial localisation functor iff it is a homotopy inverse to the relativisation functor.

5.1.2. *Overview of the equivalences of models of* $(\infty, 1)$*-categories.* We summarise the functors discussed in this section in the following diagram. For ease of legibility we only indicate the right adjoints of the Quillen equivalences. The arrow between $\mathbf{Cat}_\Delta$ and $\mathbf{RelCat}$ is dashed and points in both directions to indicate that the two categories are linked by homotopy equivalences.

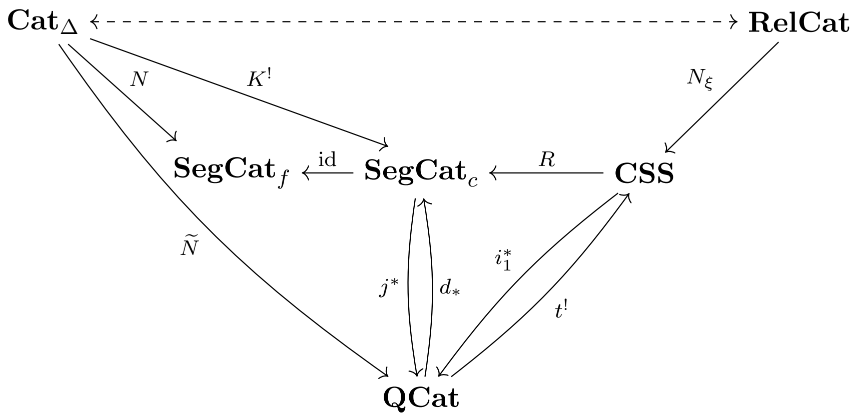


Figure 5.1.

5.2. **Summary of the homotopy theory of** $(\infty, 1)$**-categories.** Of the definitions of $(\infty, 1)$-categories we have presented in this exposition only Segal spaces satisfy all our expected requirements as they are the only structure to possess both weak composition laws and mapping spaces. Surprisingly we not only obtain an equivalent definition by considering the stricter notion of simplicial categories (analogous results are already known for bi- and tricategories as discussed in §2.3), but also by forgetting certain data of higher morphisms; we recall from §5.1 that the functor i_1^* maps Segal spaces to quasi-categories. Conversely, the functors d_* and $t^!$, being right adjoints, map quasi-categories to Reedy fibrant Segal categories and to complete Segal spaces respectively. All models have weak equivalences, namely the homotopy equivalences in Segal space, simplicial categories and quasi-categories and the weak equivalences in relative categories. Weak equivalences of relative categories should however not be considered on the same footing as they in general only form a set of *generating* weak equivalences. Indeed it is common to view relative categories as systems of generators and relations for $(\infty, 1)$-categories. One of the nice properties of model categories is that we may identify certain weak equivalences of their simplicial localisation already on the level of generating weak equivalences; both the standard simplicial localisation [DK80c, §4.1] and the hammock localisation [DK80a, §2.1] map weak equivalences between fibrant-cofibrant objects to homotopy equivalences (this follows from the respective definitions). Furthermore the morphisms between fibrant-cofibrant objects are mapped surjectively onto the path components of any simplicial localisation.

We now take a closer look at the relationship between Segal spaces and simplicial categories. Recall that the category of strict 2-categories forms a subcategory of the tricategory of bicategories and that every bicategory is biequivalent to a strict 2-category. It now suggests itself to ask whether the nerve functor $N : \mathbf{SSet\text{-}Cat} \hookrightarrow \mathbf{SSSet}$ might embed the category of categories enriched in Kan complexes in the category of Segal spaces and whether every Segal space or at least every Reedy fibrant Segal category is DK-equivalent to a category enriched in Kan complexes viewed as a bisimplicial set. Every category enriched in Kan complexes is indeed DK-equivalent to a Reedy fibrant Segal category [Joy07, Cor. 2.7] but for this very reason not every category enriched in Kan complexes can be a Segal space; this is because $j^* : \mathbf{SegCat}_c \to \mathbf{QCat}$ takes weak equivalences between Reedy fibrant Segal spaces to weak equivalences of quasi-categories by [DS95, Lemma 9.9], so if the assertion were true every quasi-category would be weakly equivalent to an object in the image of $j^* \circ N$ but these are just nerves of ordinary categories. Thus it is really the *theory* of categories enriched in Kan complexes which is the same as that of Reedy fibrant Segal categories (or any of the other definitions of $(\infty,1)$-categories)[14].

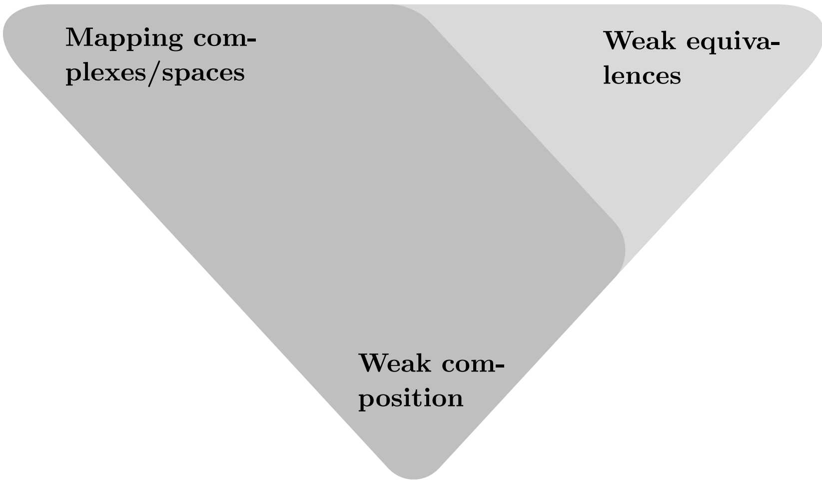


FIGURE 5.2. The main properties of the definitions of $(\infty,1)$-categories, roughly arranged as they are in Figure 5.1.

Toën provides axioms for a model category to be a model of $(\infty,1)$-categories and proves that these are satisfied by **CSS** [Toë05]. He further shows that the space of auto-equivalences of a model of $(\infty,1)$-categories is homotopy equivalent to the discrete group $\mathbb{Z}/2\mathbb{Z}$ [Toë05, Th.6.3]. The auto-equivalence not corresponding to the identity is given by sending any $(\infty,1)$-category to its opposite $(\infty,1)$-category.

6. A glance at (∞,n)-categories

The format of this section is essentially the same as that of the previous two combined: We discuss various notions of (∞,n)-categories and then compare them. Our intention in this section is however different in that we mainly want to put the theory of $(\infty,1)$-categories in perspective and thus we will make only few detailed statements. Our main reference for this section is [Ber11].

6.1. **Presheaves.** Two of the definitions we gave of $(\infty,1)$-categories, Segal spaces and quasi-categories, can be seen as adaptions of the characterisation of ordinary categories as simplicial sets (see §2.5.3). We in turn introduced the characterisation of ordinary categories as simplicial sets as an adaption of their characterisation as internal categories, by expanding the source category of the relevant presheaves from $\mathbb{G}_1$ to Δ. It is now natural to ask whether there exists a category Θ together with an ascending chain of subcategories Θ_n $(n \geq 1)$ which relate to $\mathbb{G}$ and $\mathbb{G}_n$ respectively as Δ relates to $\mathbb{G}_1$. Concretely:

(a) There exists a fully faithful functor $\mathbb{G} \hookrightarrow \Theta$ which for all $n \geq 1$ restricts to a fully faithful functor $\mathbb{G}_n \hookrightarrow \Theta_n$,
(b) there exists a fully faithful functor $\Theta \hookrightarrow \mathbf{Str}^\omega\mathbf{Cat}$, which for all $n \geq 1$ restricts to a fully faithful functor $\Theta_n \hookrightarrow \mathbf{Str}^n\mathbf{Cat}$, such that the resulting nerve functors $\mathbf{Str}^\omega\mathbf{Cat} \to \widehat{\Theta}$ and $\mathbf{Str}^n\mathbf{Cat} \to \widehat{\Theta}_n$ are fully faithful and
(c) $\Theta_1 \cong \Delta$.

[14]We are unaware in which sense the theory of strict 2-categories is the same as that of bicategories. Baez claims in [Bae97, §3.1] that the tricategory of bicategories is not triequivalent to $\mathbf{Str}^2\mathbf{Cat}$ (but does not provide a proof). Lack constructs a model structure on the category of bicategories and strict functors as well as on $\mathbf{Str}^2\mathbf{Cat}$ and shows that the two resulting model categories are Quillen equivalent [Lac04].

Joyal introduced the category Θ in [Joy97], defined as the dual category of so called finite discs. The category Θ, as desired, contains an ascending chain of categories Θ_n, now often called *n-cell categories* (e.g. in [Ara12b]). There are several equivalent characterisations of the categories Θ and Θ_n for all $n \geq 1$: Berger gives a combinatorial description of Θ in [Ber02]. In [Ber07a] he introduces the notion of the wreath product $\Delta \wr \mathcal{C}$ between Δ and any category $\mathcal{C}$ which he uses to provide an inductive definition of the categories Θ_n by specifying $\Theta_0 := 1$ [15] (the terminal category) and $\Theta_{n+1} := \Delta \wr \Theta_n$. The characterisation of the n-cell categories as wreath products makes $\Theta_1 \cong \Delta$ (point (c) above) immediate. Ara further characterises Θ by a universal property [Ara12a]. The canonical functor (a) is explained in [Ara12b, §5.5]. Point (b) was proved independently in [MZ01] and [Ber02] for $n = \infty$ and a proof for the case $n < \infty$ may be found in [Ber07a, Prop. 3.5, Th. 3.7].
As for presheaves on Δ, the presheaves on Θ_n that are isomorphic to nerves of strict n-categories are exactly those satisfying an analogue of the Segal condition [Ber02, Rmk. 1.13] or equivalently satisfying a unique horn filling condition [Ber02, Prop. 1.12].

6.1.1. *Quasi-n-categories and n-quasi-categories.* Joyal's motivation for introducing the category Θ was to define a notion of weak ω-category as presheaves on Θ satisfying a non-unique inner horn filling condition. At the end of [Ber11, §4.2] it is announced that Barwick has defined a notion of *quasi-n-categories* in this sense, however we have not been able to find any further references for this theory.
Ara has defined a model structure on the categories of presheaves on Θ_n for all $n \geq 1$ based on an idea of Joyal and Cisinski (see [Joy08a, §45]) which coincides with the model structure on **QCat** for $n = 1$ [Ara12b]. We denote these model categories by $\mathbf{Q}^n\mathbf{Cat}$. He then defines *n-quasi-categories* to be the fibrant objects in $\mathbf{Q}^n\mathbf{Cat}$. The nerves of strict n-categories are however *not* in general n-quasi-categories [Ara12b, Rmk. 5.27].

6.1.2. Θ_n*-spaces.* We motivated the notion of Segal spaces by noting that if categories may be characterised as certain simplicial objects in **Set** then it is reasonable to believe that $(\infty, 1)$-categories should be able to be defined as certain simplicial objects in **SSet**. This line of thought may then be extended to the n-dimensional case and it is natural to suspect that (∞, n)-categories should be able to be defined as simplicial presheaves on Θ_n, that is, as functors $\Theta_n^{\mathrm{op}} \to \mathbf{SSet}$. Rezk provides just such a definition in [Rez10], namely that of Θ_n*-space*. It can be shown that Θ_n is a Reedy-category and that the Reedy model structure on the category of simplicial presheaves on Θ_n again coincides with the injective model structure [BR13b].
A Θ_n-space is then a Reedy fibrant object in the category of presheaves on Θ_n satisfying a Segal condition[16] as well as a certain completeness condition. The Reedy model category structure on the category of simplicial presheaves on Θ_n may again be localised to yield a model category in which the fibrant objects are Θ_n-spaces. For $n = 1$ we recover the model category **CSS**.

6.1.3. *Other approaches using presheaves.* There are several further proposed definitions of higher categories using presheaves.
Ayala and Rozenblyum sketch a theory of simplicial presheaves on stratified manifolds to model "(∞, n)-categories with adjoints" [Aya12]; they relate this model to that of Rezk's Θ_n-spaces.
Street introduced the notion of *complicial sets* in [Str87]; these are certain *stratified sets*, i.e. pairs (X, tX) consisting of a simplicial set X together with a subset tX of its simplices satisfying various conditions, which furthermore must satisfy a certain horn-filling condition. Verity proves a conjecture of Street that the category of strict ω-categories is equivalent to that of complicial sets [Ver05]. He further expands the theory of complicial sets to that of weak complicial sets, characterised by weakening the horn-filling conditions and constructs a model category structure on the category of stratified sets such that the fibrant objects are exactly the weak complicial sets [Ver08], [Ver07].
Baez and Dolan introduced the category of *opetopes* and an n-category is then a presheaf on

[15] In [Ber02] and [Ara12a], [Ara13] Θ_0 denotes a different category closely related to Θ.
[16] We do not know how this Segal condition relates to the Segal condition in [Ber02, Rmk. 1.13]!

the category of opetopes, which must again satisfy certain filler conditions (in this setting the shapes to be filled are called 'niches') [BD98].

6.2. **Weak enrichment.** In our discussion of strict n-categories in §2.1.1 we mentioned that sometimes for a category $\mathcal{V}$ the category $\mathcal{V}$-**Cat** is equivalent to a subcategory of $\mathrm{Cat}(\mathcal{V})$. Notably strict $(n+1)$-categories may be defined as certain internal categories in $\mathbf{Str}^n\mathbf{Cat}$. We may view Segal spaces as internal $(\infty,1)$-categories in the $(\infty,1)$-category of $(\infty,0)$-categories and therefore it is natural to suspect that given a definition of (∞,n)-categories, $(\infty,n+1)$-categories may be defined as certain simplicial objects in the $(\infty,1)$-category of (∞,n)-categories.
Given an $(\infty,1)$-category $\mathcal{C}$ Lurie defines a *category object* of $\mathcal{C}$ to be a simplicial objet in $\mathcal{C}$ satisfying a Segal condition [Lur09b, Def. 1.1.1]. Like for Segal spaces we must impose further conditions on the category objects in $\mathcal{C}$ in order to have a good theory. In the theories we know of one either generalises the discreteness condition of Segal categories or the completeness condition of complete Segal spaces.

6.2.1. *Weakly enriched categories.* Pellisier introduced the notion of weakly enriched categories in his thesis [Pel02]. The $(\infty,1)$-categories in which Pellisier considers category objects are Cartesian model categories $\mathcal{M}$ satisfying certain conditions, notably the colimit of the constant functor from any set, considered as a category, to the final object $*$ in $\mathcal{M}$ must exist and the resulting functor $\mathbf{Set} \to \mathcal{M}$ must be fully faithful [Sim12, Ch. 6]. Objects isomorphic to an object in the image of this functor are called *discrete*. A simplicial object X in $\mathcal{M}$ is called an $\mathcal{M}$-*precategory* if X_0 is discrete and a *weak* $\mathcal{M}$-*category* if all the Segal maps are weak equivalences. The category of $\mathcal{M}$-precategories may be endowed with different model structures which possess various properties depending on the properties of $\mathcal{M}$ [Sim12, Ch. 19]. Lurie considers the more general notions of $\mathcal{M}$-*enriched preSegal category* and $\mathcal{M}$-*enriched Segal category* in an arbitrary model category $\mathcal{M}$[Lur09b, §1.3]. These notions coincide with those of $\mathcal{M}$-precategory and weak $\mathcal{M}$-category if $\mathcal{M}$ satisfies the conditions sketched above [Sim12, Ch. 6].
Starting with a suitable model category of 0-categories, general n-categories may be obtained by iterating Pellisier's or Lurie's construction. Choosing the model of 0-categories to be the model category **Set**, with the weak equivalences being the bijections and all maps being fibrations and cofibrations, we obtain Tamsamani's *n-nerves* [Tam99], [Sim12, §§20.2.1 - 20.2.6]. Starting with the model of $(\infty,0)$-categories **SSet** we obtain the *Segal n-categories* which were introduced by Hirschowitz and Simpson in [HS98]. We denote the model category of Segal n-categories by $\mathbf{Seg}^n\mathbf{Cat}$.
Bergner and Rezk consider weak $\Theta_{n-1}\mathbf{Sp}$-categories, which they call *Segal category objects in* $\Theta_{n-1}\mathbf{Sp}$ [BR13a]. There are two model structures, analogous to those on the category of Segal precategories. We denote the two categories by $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_c$ and $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_f$.

6.2.2. *Complete Segal space objects.* Lurie determines a class of quasi-categories called *absolute distributors* in which it is possible to define category objects satisfying a completeness condition [Lur09b, Ch. 1]. A simplicial model category $\mathcal{M}$ is likewise called an absolute distributor if it satisfies some technical properties and if the homotopy coherent nerve of the full simplicial subcategory or $\mathcal{M}$ spanned by the fibrant-cofibrant objects is an absolute distributor [Lur09b, §1.5]. The category of simplicial objects in an absolute distributor may be endowed with a model structure such that it is again an absolute distributor and such that the fibrant objects are exactly the complete Segal space objects [Lur09b, Prop. 1.5.4]. We denote the category of simplicial objects in an absolute distributor $\mathcal{M}$ with this model structure by $\mathbf{CSS}(\mathcal{M})$. The category **SSet** with the Quillen model structure is an absolute distributor. Setting $\mathbf{CSS}^0 := \mathbf{SSet}$ Lurie iterates the complete Segal object construction to define *n-fold Segal spaces* which corresponding model category is denoted by $\mathbf{CSS}^n$.
In [Ber11, p. 26] Bergner claims that there is a model structure on the category of simplicial objects in $\Theta_n\mathbf{Sp}$ such that the fibrant objects satisfy Segal and completeness conditions; we are unaware whether these are complete Segal space objects in the sense of Lurie.

6.3. **Strict enrichment.** Going from (∞, n)-categories to $(\infty, n+1)$-categories may sometimes be achieved by strict enrichment. If a monoidal model category $\mathcal{M}$ satisfies appropriate conditions it is possible to endow the category $\mathcal{M}$-**Cat** with a model structure [Lur09a, A.3]. Applying this construction to the Cartesian model category **SSet** produces the model category $\mathbf{Cat}_\Delta$. The model category on $\mathcal{M}$-**Cat** is in general not Cartesian so we may neither continue enriching strictly nor weakly.
Bergner and Rezk consider categories strictly enriched in $\Theta_n\mathbf{Sp}$ [BR13a, §3].

6.4. **n-relative categories.** Barwick and Kan introduced the notion of *n-relative categories* in [BK11] which generalises that of relative categories (see §4.1.2); these are $(n+2)$-tuples consisting of a category $\mathcal{C}$ together with $n+1$ subcategories, each containing all objects of $\mathcal{C}$ and satisfying certain rather intricate relations with respect to each other. The category of n-relative categories is denoted $\mathbf{Rel}^n\mathbf{Cat}$. Barwick and Kan then generalise the nerve functor $N_\xi : \mathbf{RelCat} \to \mathbf{CSS}$ to a functor $\mathbf{Rel}^n\mathbf{Cat} \to \mathbf{CSS}^n$ and they again lift the model structure of $\mathbf{CSS}^n$ to $\mathbf{Rel}^n\mathbf{Cat}$. As a special case we get $\mathbf{Rel}^1\mathbf{Cat} \cong \mathbf{RelCat}$ and $\mathbf{Rel}^0\mathbf{Cat} \cong \mathbf{Cat}$ where the model structure induced on **Cat** is the one considered by Thomason in [Tho80].

6.5. **The homotopy theory of (∞, n)-categories.** Before turning to the Quillen equivalences linking the various models of (∞, n)-categories we discuss the axiomatic approach to the homotopy theory of (∞, n)-categories introduced by Barwick and Schommer-Pries [BS11]. Unlike the axioms proposed by Toën in [Toë05] the underlying $(\infty, 1)$-category of the $(\infty, n+1)$-category of (∞, n)-categories is not given as a model category but as a quasi-category.
They generalise the theorem of Toën characterising the space of auto-equivalences of a theory of (∞, n)-categories; it is shown that it is homotopy equivalent to the discrete group $(\mathbb{Z}/2\mathbb{Z})^n$. For every ascending sequence of integers $0 < i_1 < \cdots < i_K < n$ an auto-equivalence of a model of (∞, n)-categories is given by sending any (∞, n)-category to the corresponding (∞, n)-category which is opposite on the level of i_k-morphisms for $0 < k < K$ [BS11, Lemma 4.5, Th. 8.13].
Furthermore it is shown that the $(\infty, 1)$-category of Θ_n-spaces [BS11, Ch. 11] as well as that of n-fold Segal spaces [BS11, Ch. 12] satisfy their axioms and it is stated that this is true of n-relative categories [BS11, Ex. 13.3].
We denote by $N^{\mathrm{H}} : \mathbf{RelCat} \to \mathbf{QCat}$ the functor obtained by composing the hammock localisation functor $L^{\mathrm{H}} : \mathbf{RelCat} \to \mathbf{Cat}_\Delta$, functorial fibrant replacement in $\mathbf{Cat}_\Delta$ and the homotopy coherent nerve functor $\widetilde{N} : \mathbf{Cat}_\Delta \to \mathbf{QCat}$. Barwick and Schommer-Pries call a model category $\mathcal{M}$ a *model category of (∞, n)-categories* if $N^{\mathrm{H}}\mathcal{M}$ satisfies their axioms. They then state a proposition which allows them to recognise Quillen adjunctions between model categories of (∞, n)-categories which are Quillen equivalences by considering their image under N^{H} [BS11, Prop. 13.10].

6.6. **Some Quillen equivalences between models of (∞, n)-categories.** A Quillen equivalence analogous to the one between $\mathbf{Cat}_\Delta$ and $\mathbf{SegCat}_f$ is possible between $\mathcal{M}$-**Cat** and the category of weak $\mathcal{M}$-categories for a general class of model categories $\mathcal{M}$ [Sim12, §19.5], [Lur09b, Th. 2.2.16]. In [BR13a, §3] a Quillen equivalence is constructed between $\Theta_{n-1}\mathbf{Sp}$-**Cat** and $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_f$. Furthermore the identity functor gives a Quillen equivalence between $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_f$ and $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_c$ [BR13a, Prop. 5.8].
It is conjectured that there is a Quillen equivalence between $\mathbf{CSS}(\Theta_{n-1}\mathbf{Sp})$ and $\mathbf{Seg}(\Theta_{n-1}\mathbf{Sp})_c$, similar to the one between **CSS** and $\mathbf{SegCat}_c$ [BR13a, §1.1]. Lurie identifies a general class of model categories $\mathcal{M}$ such that the inclusion $\mathbf{Seg}(\mathcal{M})_c \hookrightarrow \mathbf{CSS}(\mathcal{M})$ is the left adjoint of a Quillen equivalence [Lur09b, Prop. 2.3.1]; we are unaware whether $\Theta_n\mathbf{Sp}$ is such a model category.
A direct Quillen equivalence between $\mathbf{CSS}^n$ and $\mathbf{Seg}^n\mathbf{Cat}$ seems to be at hand. Given a pair of model categories $\mathcal{M}$ and $\mathcal{N}$ satisfying appropriate conditions as well as a Quillen equivalence $\mathcal{M} \rightleftarrows \mathcal{N}$, a theorem implying a Quillen equivalence $\mathbf{Seg}(\mathcal{M})_c \rightleftarrows \mathbf{Seg}(\mathcal{N})_c$ or $\mathbf{CSS}(\mathcal{M}) \rightleftarrows \mathbf{CSS}(\mathcal{N})$ and a verification that $\mathbf{CSS}^{n-1}$ or $\mathbf{Seg}^{n-1}\mathbf{Cat}_c$ satisfies the conditions in [Lur09b, Prop. 2.3.1] respectively would suffice to establish that the canonical inclusion

$\mathbf{Seg}^n\mathbf{Cat}_c \hookrightarrow \mathbf{CSS}^n$ is the left adjoint of a Quillen equivalence because we could then either compose the top or the bottom left Quillen functors in the following diagram:

$$\begin{array}{ccc} \mathbf{Seg}(\mathbf{Seg}^{n-1}\mathbf{Cat}_c)_c & \longrightarrow & \mathbf{Seg}(\mathbf{CSS}^{n-1})_c \\ \downarrow & & \downarrow \\ \mathbf{CSS}(\mathbf{Seg}^{n-1}\mathbf{Cat}_c) & \longrightarrow & \mathbf{CSS}(\mathbf{CSS}^{n-1}). \end{array}$$

It is also conjectured that there exists a Quillen equivalence between $\mathbf{CSS}(\Theta_{n-1}\mathbf{Sp})$ and $\Theta_n\mathbf{Sp}$ [Ber11, Conj. 6.3] induced by the diagonal functors $\Delta \times \Theta_{n-1} \to \Theta_n$ [Ber07a, Def. 3.8]. Analogously as for the categories $\mathbb{G}_n$ in (2.1) we obtain a chain of functors

$$\Delta \times \overset{n\times}{\cdots} \times \Delta \;\to\; \Delta \times \overset{n-2\times}{\cdots} \times \Delta \times \Theta_2 \;\to\; \cdots \;\to\; \Delta \times \Theta_{n-1} \;\to\; \Theta_n$$

which should induce a chain of Quillen equivalences

$$\mathbf{CSS}^n \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{CSS}^{n-1}(\Theta_2\mathbf{Sp}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \cdots \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \mathbf{CSS}(\Theta_{n-1}\mathbf{Sp}) \underset{\longleftarrow}{\overset{\perp}{\longrightarrow}} \Theta_n\mathbf{Sp}.$$

A proof is expected in the upcoming paper [BR]. Barwick and Schommer-Pries claim in [BS11, Ex. 13.12] that the composition of the above adjoints is a Quillen equivalence between $\mathbf{CSS}^n$ and $\Theta_n\mathbf{Sp}$ by applying [BS11, Prop. 13.10] (discussed in §6.5). If it can be shown that $\Theta_n\mathbf{Sp}$ is an absolute distributor [Lur09b, Def. 1.3.3] for all $n \geq 1$, then by [BS11, Ex. 13.7], [Lur09b, Corr. 1.3.4] and [BS11, Prop. 13.10] it would be verified that the intermediate adjunctions are also Quillen equivalences.
Finally, Ara has constructed Quillen equivalences between $\Theta_n\mathbf{Sp}$ and $\mathbf{Q}^n\mathbf{Cat}$ in both directions which restrict to those constructed by Joyal and Tierney for $n = 1$. Notably, if we view simplicial presheaves on Θ_n as presheaves on $\Theta_n \times \Delta$, the functor which generalises i_1^*, is again given by sending an object $X \in \Theta_n\mathbf{Sp}$ to $X_{\bullet 0}$ [Ara12b, §2.15, Th. 7.4].

6.6.1. *Overview of the equivalences of models of (∞, n)-categories.* We summarise the Quillen adjunctions discussed here in the following diagram; as in Figure 5.1 we only depict their right adjoints. The functors labled with a question mark are those for which it is only conjectured that they form the right adjoint of a Quillen equivalence.

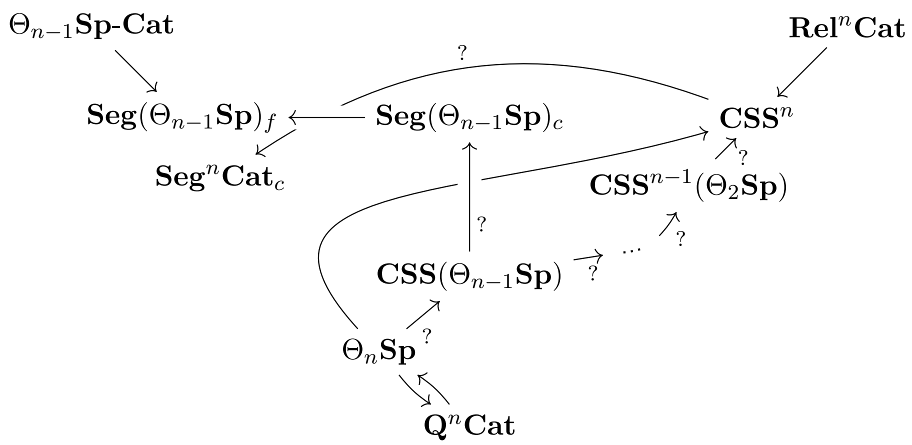


FIGURE 6.3.

6.7. **Reviewing $(\infty, 1)$-categories.** Looking back at $(\infty, 1)$-categories we may now reinterpret certain facts. As categories strictly enriched in (∞, n)-categories are generally not Cartesian, enriching in this way is something which can only be done at the "top level". For simplicial categories we simply choose this top level to be $(\infty, 0)$-categories. We may view 1-fold internal categories both as presheaves on $\mathbb{G}_n$ or on the n-fold product $\mathbb{G}_1 \times \cdots \times \mathbb{G}_1$ with $n = 1$. The analogous interpretation holds for Segal spaces with Θ_n and the n-fold product $\Delta \times \cdots \times \Delta$ with $n = 1$. However, only complete Segal spaces seem to extend to higher versions in both ways, as Θ_n-spaces and as n-fold complete Segal spaces; the only higher versions of Segal categories we know of are given by simplicial presheaves on $\Delta \times \cdots \times \Delta$. Finally we may again obtain an equivalent definition of (∞, n)-categories by forgetting certain data of Θ_n-spaces. The notion that weak composition is enough to define (∞, n)-categories seems doubtful to us as the fibrant objects in $\mathbf{Q}^n\mathbf{Cat}$ do not coincide with objects satisfying any reasonable horn-filling conditions. Toën notes that the fact that any relative category generates an $(\infty, 1)$-category accounts for their relative abundance compared to (∞, n)-categories [Toë05, p. 234]. The intricate definition of relative n-categories amplifies the pertinence of this observation.

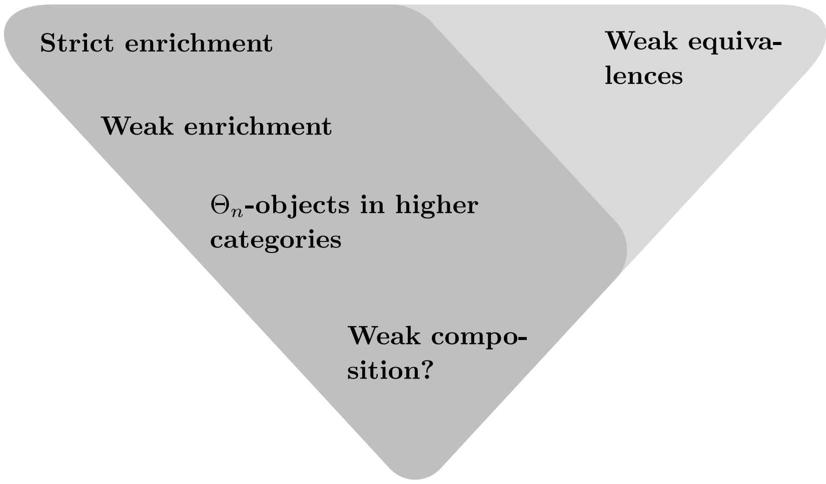


Figure 6.4. The main properties of the definitions of (∞, n)-categories, roughly arranged as they are in Figure 6.3.

6.8. **The $(\infty, n+1)$-category of (∞, n)-categories.** At least three of the models for (∞, n)-categories we have considered, Θ_n-spaces, n-quasi-categories and Segal n-categories, are known to be Cartesian. At least in **CSS** the internal hom-object between two complete Segal spaces is again a complete Segal space [Rez01, Corr. 7.3]; the analogous question for Segal n-categories is discussed in [Sim12, Ch. 20.5]. In all these cases we could construct the $(\infty, n+1)$-category of (∞, n)-categories via internal hom to obtain a category strictly enriched in the respective models of (∞, n)-categories. We find it natural to ask whether these are the "correct" $(\infty, n+1)$-categories of (∞, n)-categories. This might be the starting point for the comparison of various $(\infty, n+1)$-categories of (∞, n)-categories according to different definitions similarly as for strict n-categories in Remark 2.1.3.

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