On Shalom–Tao's Non-Quantitative Proof of Gromov's Polynomial Growth Theorem

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1 Towards the statement of Gromov's theorem

Definition. Let (G, S) be a finitely generated group (S generating set, $S^{-1} = S$, $e \notin S$). We define

$$d_S(g, h) := \min \{k \in \mathbb{N}_0 | \, \exists s_1, \ldots, s_k \in S \colon \, h = gs_1 \cdots s_k\}.$$

- (G, d_S) is a metric space; d_S is left-invariant, so G acts on itself from the left by isometries.
- (G, d_S) is discrete, but we can realize it as the (combinatorial) Cayley graph $\Gamma_S(G)$ of G. The group G acts on $\Gamma_S(G)$ by isometries.
- d_S depends highly on the generating set S. However, if T, S are generating sets of G, then $\mathrm{id}_G : (G, d_S) \longrightarrow (G, d_T)$ is a K-Lipschitz map, with $K = \max_{s \in S} d_T(e, s)$. Particularly, they are quasi-isometric.
- Notation: $B_S(g, r)$ closed ball at g, radius r; $B_S(r) := B_S(g, r)$. Notice: $\#B_S(g, r) = \#B_S(r)$ for all g.

Definition. Let (G, S) be as above. The *growth function* $\beta_{G,S}$ of (G, S) is the map $n \longmapsto \#B_S(n)$. Notice that the growth function of a group (G, S) has an exponential bound given by

$$\beta_{G,S}(n) \leq \sum_{i=0}^{n} |S|^i,$$

since the amount of elements in $B_S(n)$ is at most the amount of words of length n in the generators.

Example. Three examples of easy computation:

- Consider $\mathbb{Z}$ with $S = \{-1, 1\}$ and $T = \{-3, -2, 2, 3\}$. Then

$$\beta_{\mathbb{Z},S}(n) = 2n + 1 \quad \text{ and } \quad \beta_{\mathbb{Z},T}(n) = \begin{cases} 1, & n = 0 \\ 5, & n = 2 \\ 6n + 1, & n \geq 2. \end{cases}$$

- Let $\mathbb{F}_2$ be the free group of rank 2, and $S = \{a, b, a^{-1}, b^{-1}\}$. Then $\beta_{\mathbb{F}_2,S}(n) = 2 \cdot 3^n - 1$.
- Let $\mathbb{Z}^2$ be the free abelian group of rank 2 and let $S = \{e_1, e_2, -e_1, -e_2\}$, where e_1, e_2 are the canonical unit vectors. Then, $\beta_{\mathbb{Z}^2,S}(n) = 2n^2 + 2n + 1$.

Definition. Let (G, S) be a finitely generated group.

1. G has *polynomial growth* if there exist constants $c > 0$, $d \in \mathbb{N}_0$ such that $\beta_{G,S}(n) \le cn^d$ for all n (G has polynomial growth of degree $\le d$).
2. G has *exponential growth* if there exist constants $c > 0$, $v > 1$ such that $\beta_{G.S}(n) > cv^n$ for all n.
3. G has *intermediate growth* if it has neither polynomial nor exponential growth.

Fact. Having polynomial growth of degree $\le d$ is an invariant of quasi-isometry (and in general, the "growth class" of a group (with respect to the weak equivalence relation on growth functions[1]) is a quasi-isometry-invariant).

Fact. Every finitely generated group has a growth class in one of these disjoint types. (However, there are growth functions that are not comparable neither to polynomials nor to exponential functions, e.g. $\exp(n^{\sin n})$ fluctuates between 1 and e^n.)

Some motivation: The study of growth of groups is related to:

1. the volume growth of Riemannian manifolds:

   Proposition (Švarc–Milnor)**.** Let X be a complete Riemannian manifold with Riemannian distance d, and $x_0 \in X$ a point. For each $t \ge 0$, denote $v(t)$ the Riemannian volume of the closed ball of center x_0 and radius t in X. Let Γ be a group acting properly and co-compactly by isometries on X (hence, finitely generated), and let β denote its growth function with respect to some set of generators. Then, v and β are equivalent as growth functions. Particularly, this applies if X is the universal covering of a compact manifold and $\Gamma = \pi_1(X)$.

2. the curvature of Riemannian manifolds:

   Proposition (Milnor '68)**.** Let X be a compact Riemannian manifold of strictly negative sectional curvature. Then, the fundamental group of X is of exponential growth.

   Proposition (Milnor '68)**.** The fundamental group of a compact Riemannian manifold X with mean curvature ≥ 0 is of polynomial growth, with $\beta_{\pi_1(X)}(k) \preceq k^{\dim X}$.

3. amenability of groups.

   Definition. A finitely generated group (G, S) is amenable if it has a Følner sequence, i.e. a sequence $(F_k)_{k\ge 0}$ of finite subsets of G such that

   $$\lim_{k\to\infty} \frac{|F_k \cup \partial_S F_k|}{|F_k|} = 1,$$

   where $\partial_S A = \{g \in G |\, g \notin A \text{ and } gs \in A \text{ for some } s \in S\}$ denotes the S-boundary of a subset $A \subset G$.

   Proposition. If G is of subexponential growth, then there exists a subsequence of $(B_S(k))_k$ that is a Følner sequence of G (if it has polynomial growth $(B_S(k))_k$ is the Følner sequence). So, all groups of subexponential growth are amenable.

[1] Let $f, f' : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be growth functions (non-decreasing). We say $f \preceq f'$ ("f is weakly dominated by f'") if there exist K, L constants such that $f(n) = Kf'(Kn + L) + L$ for all n. Moreover, we say $f \approx f'$ ("f and f' are weakly equivalent") if $f \preceq f' \preceq f$.

Example (Exponential growth).

- Free groups (non-amenable).
- $H = \left\langle \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle \leqslant \mathrm{GL}(n, \mathbb{R})$ has exponential growth (see [2, Chapter VII, Example 3]).
- Every surface group Γ_g with $g \geq 2$ (i.e. fundamental group of the compact orientable surface Σ_g of genus g). The surface Σ_g for $g \geq 2$ can be provided with a metric of negative curvature by regarding it as a quotient $\mathbb{H}^2/\Gamma$, where $\Gamma \leqslant \mathrm{Iso}\left(\mathbb{H}^2\right)^0$ is discrete cocompact. By a proposition above, $\Gamma_g = \pi_1(\Sigma_g)$ has exponential growth.

Example (Intermediate growth). Milnor (1968) posed a question on the existence of examples in this class, as none were known at that point. Grigorchuk (1984) constructed such a group (subgroup of the automorphism group $\mathrm{Aut}(T_2)$ of the infinite regular binary rooted tree T_2), answering this question affirmatively. All of the examples found so far in this class are somehow related to this one, and grow faster than $e^{\sqrt{n}}$.

Example (Polynomial growth). The following kinds of finitely generated groups have polynomial growth

- *Finite groups* (growth bounded by the order; have degree 0).
- *Abelian groups* [for example, $\mathbb{Z}^d$]: If (G, S) is a finitely generated abelian group with generating set $S = \{s_1, \ldots, s_d\}$, then

$$\begin{aligned} \# B_S(r) &= \# \left\{ s_1^{\varepsilon_1} \cdots s_d^{\varepsilon_d} \,|\, \varepsilon_i \in \mathbb{N}_0, \sum \varepsilon_i \leq r \right\} \leq \# \left\{ (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}_0^d \,|\, \sum \varepsilon_i \leq r \right\} \\ &\leq \# \left\{ (\varepsilon_1, \ldots, \varepsilon_d) \in \mathbb{N}_0^d \,|\, 0 \leq \varepsilon_i \leq r \right\} = (r+1)^d \leq c r^d \end{aligned}$$

  for some c.

- *Nilpotent groups* [for example, the discrete Heisenberg group]: (Milnor, Wolf; Bass, Guivarc'h; others...) The proof goes by induction on the nilpotency class of the group, and uses the following two lemmas.

  Lemma. Let G be a finitely generated nilpotent group. Then, the commutator group $G_1 = [G, G]$ is finitely generated.

  Lemma. Let G be a nilpotent group and H a finitely generated subgroup of polynomial growth of G containing G_1. Let also $t \in G$. Then $K := \langle H \cup \{t\} \rangle$ has polynomial growth.

- *Virtually nilpotent groups*[2] [for example, semidirect products of a finite and a nilpotent group; contains both classes] The proof is a direct consequence of the nilpotent case and the following observation.

  Lemma. Let G be a finitely generated group. If H is a finite-index subgroup of G, then H is finitely generated. If in addition H has polynomial growth, then so does G.

This last example is the "universal" example:

Theorem (Gromov '81). *Every finitely generated group of polynomial growth is virtually nilpotent.*

[2] A group is virtually *blah* if it has a finite-index *blah* subgroup.

2 Some words on the existing proofs

There exist (at least) three different proofs of Gromov's polynomial growth theorem: Gromov's original proof, Kleiner's and Shalom-Tao. All of these proofs have one philosophy in common:

Goal: To construct a finite-index subgroup Δ of our finitely generated group of polynomial growth G that can be mapped homomorphically onto $\mathbb{Z}$.

After this, we go by induction on the degree of polynomial growth of G: notice that Δ must have the same degree of growth as G; one can show that the kernel of a map $\Delta \twoheadrightarrow \mathbb{Z}$ is finitely generated and has polynomial growth of degree strictly less than the one of G; by induction, Δ must have a finite-index nilpotent subgroup and "therefore" so does G.

Now, let us discuss briefly what elements are particular of each proof.

Gromov's original proof (1981):

This proof is beautiful (but complicated) and of geometric nature. It was the first application of the Montgomery–Zippin–Yamabe structure theory of locally compact groups, which gives solution to Hilbert's fifth problem.

Gromov shows that a subsequence of the sequence $X_n = \left(G, \frac{1}{n}d\right)$ of metric spaces converges (as a Gromov–Hausdorff limit) to a space X_∞ with the following properties:

- X_∞ is homogeneous (for any two points, there is an isometry that takes one point to another).
- X_∞ is connected and locally connected.
- X_∞ is complete.
- X_∞ is locally compact and finite (Hausdorff) dimensional.

Because of Montgomery–Zippin, the isometry group $\mathrm{Iso}(X_\infty)$ is a Lie group with finitely many connected components. This fact, together with a theorem of Jordan's and the Tits' alternative for linear groups, allows the construction of the seeked finite-index subgroup Δ and the epimorphism $\Delta \twoheadrightarrow \mathbb{Z}$.

Kleiner's proof (2007):

This is a more analytic proof, and it is based on a weaker form of a former result by Colding–Minicozzi (1997) that guarantees the finite-dimensionality of the space of harmonic functions on the group G. The original theorem by Colding–Minicozzi relies on Gromov's theorem, while Kleiner's proof is independent of it. The epimorphism above arises from this fact (the "how" will be explained later; the argument is the same in Shalom–Tao's proof).

It does not make use Montgomery–Zippin–Yamabe, but it does rely on the Tits' alternative for linear groups (or on a result by Shalom: amenable linear groups are virtually solvable). An important element of the proof of the finite-dimensionality is a new Poincaré inequality shown by Kleiner in the same paper (bound of a function in terms of its derivatives and the geometry its domain of definition).

Shalom–Tao's proof (2009-2010):

There are two versions by Shalom–Tao. The first one (*not* the topic of our talk) is "finitary", in the sense that it keeps track of the degree of polynomial growth of the group (and many other constants!) to give a bound in the nilpotency class of the group. The strategy of the proof is the same as Kleiner's (i.e. exploiting properties of the space of harmonic functions on G), but "making the proof of the statements (...) as elementary as possible, so that they may be made quantitative". Because of these many quantifications, the argument turns out to be quite lengthy. One main difference with Kleiner's proof, though, is the avoidance of Tits' alternative (or Shalom's theorem).

The second version of the proof, published in Tao's blog, is a variation of the last one that disregards all the quantifications. The non-usage of Tits' alternative makes it even more elementary than Kleiner's. We'll go for this one.

3 The non-quantitative proof by Shalom–Tao

Before going to the proof, we give some central definitions:

Definition. Let $(G,\, S)$ be a finitely generated group and $\mathbb{F}$ denote the field of real or complex numbers.

1. A function $f : G \longrightarrow \mathbb{F}$ is said to be *Lipschitz with respect to* S (or *Lipschitz on* (G, S)) if there exists a constant $C \geq 0$ such that $|f(x) - f(xs)| \leq C$, for every $x \in G$ and every $s \in S$. In this case, we denote by $\mathrm{LIP}(f)$ the infimum of all such non-negative constants, or, equivalently

$$\mathrm{LIP}(f) := \sup_{x\in G,\, s\in S} |f(x) - f(xs)| \,.$$

   The reader can verify that the $\mathbb{F}$-valued Lipschitz functions on (G, S) form an $\mathbb{R}$-vector space (subspace of $\mathbb{F}^G$); we shall denote it by $\mathrm{Lip}(G, S; \mathbb{F})$. Moreover, the map LIP (as above) is a seminorm on $\mathrm{Lip}(G, \mathbb{F})$ that vanishes only on the constant functions. We also remark that all bounded functions on (G, S) are automatically Lipschitz.

2. The *mean-value operator* $\mathcal{M}_{G,S} : \mathbb{F}^G \longrightarrow \mathbb{F}^G$ on $\mathbb{F}$-valued functions on (G, S) is defined by

$$\mathcal{M}_{G,S} f(x) := \frac{1}{|S|} \sum_{s\in S} f(xs) \,,$$

   for every $x \in G$. We define now the *Laplacian operator* $\Delta_{G,S} := \mathrm{id}_{\mathbb{F}^G} - \mathcal{M}_{G,S}$. A function $f : G \longrightarrow \mathbb{F}$ is said to be *harmonic with respect to* S (or *harmonic on* (G, S)) if $\Delta_{G,S} f \equiv 0$, that is, if the mean-value property

$$f(x) = \frac{1}{|S|} \sum_{s\in S} f(xs)$$

   holds, for every $x \in G$. The set of all $\mathbb{F}$-valued harmonic functions on (G, S) is an $\mathbb{R}$-vector space (subspace of $\mathbb{F}^G$), which we shall denote $\mathcal{H}(G, S; \mathbb{F})$.

3. We denote by $\mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{F})$ the $\mathbb{R}$-vector space of $\mathbb{F}$-valued Lipschitz harmonic functions on (G, S).

Remark.

- Whenever it does not lead to confusions, we will omit the subindex "G, S" in the notations for the mean-value and the Laplacian operator.
- The restriction of the Laplacian operator to the Hilbert space $\ell^2(G)$ is of particular interest. Indeed, it can be easily checked that $\Delta : \ell^2(G) \longrightarrow \ell^2(G)$ is a (well defined) positive definite self-adjoint bounded linear operator, with operator norm≤ 2.
- Liouville's theorem holds for harmonic functions on (G, S), that is, every (real- or complex-valued) bounded harmonic function on (G, S) is constant.
- $\mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{C}) = \mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{R}) + i\mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{R})$.

Let us prove now the main theorem by reducing the problem to the following four propositions:

Proposition A. Let (G, S) be an infinite finitely generated group. Then, there exists a non-constant Lipschitz harmonic function $f : G \longrightarrow \mathbb{R}$.

Proposition B (**Kleiner**). The vector space $\mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{R})$ on a group (G, S) of polynomial growth is finite-dimensional.

Proposition C (**Gromov's theorem for compact Lie groups**). Let G be a finitely generated subgroup of a compact linear (real) Lie group $H \leqslant \mathrm{GL}_n(\mathbb{C})$ of polynomial growth. Then, G is virtually abelian.

Proposition D (**"Algebraic lemma"**). For a fixed $d \in \mathbb{N}_0$, assume that all groups of polynomial growth of degree $\leq d-1$ are virtually nilpotent, and let (G, S) be a group of polynomial growth of degree d. If G contains a finite-index subgroup that can be mapped homomorphically onto $\mathbb{Z}$, then G is virtually nilpotent.

Proof of the main theorem. Let G be a group of polynomial growth of degree $d \in \mathbb{N}_0$, and fix a symmetric finite generating set S of G. The proof goes by induction on d. The base case $d = 0$ is trivial, as this implies the finiteness of the group and virtual nilpotence follows.

Assume now that the theorem holds for groups with polynomial growth order of at most $d-1$, with $d > 0$. The fact that the degree of growth d is non-zero implies that the group G is infinite. Let $\mathcal{V}$ denote the space $\mathcal{H}^{\mathrm{Lip}}(G, S; \mathbb{C})$, which by Proposition A contains $\mathbb{C}$ properly. The reader can verify that the group G acts on $\mathcal{V}$ by linear automorphisms, where the action is given by

$$\gamma \cdot f : G \ni x \longmapsto f\left(\gamma^{-1}x\right) \in \mathbb{C},$$

for all $\gamma \in G$. This action induces a canonical G-action on the quotient $\mathcal{V}/\mathbb{C}$:

$$\begin{array}{cccl} \Psi : & G & \longrightarrow & \mathrm{GL}\left(\mathcal{V}/\mathbb{C}\right) \\ & \gamma & \longmapsto & \left[f + \mathbb{C} \longmapsto \gamma \cdot (f + \mathbb{C}) := \gamma \cdot f + \mathbb{C}\right]. \end{array}$$

Notice that the Lipschitz seminorm on $\mathcal{V}$ turns into an actual norm on the quotient $\mathcal{V}/\mathbb{C}$ (which we shall denote $\|\cdot\|_{\mathrm{LIP}}$), and that this norm is preserved by the action Ψ, that is,

$$\|\Psi(\gamma)(f + \mathbb{C})\|_{\mathrm{LIP}} = \|\gamma \cdot (f + \mathbb{C})\|_{\mathrm{LIP}} = \|f + \mathbb{C}\|_{\mathrm{LIP}},$$

for all $\gamma \in G$ and all $f \in \mathcal{V}$. This can be rephrased as follows: if we consider the operator norm on $\operatorname{End}(\mathcal{V}/\mathbb{C})$ (induced by the Lipschitz norm), then G' is a bounded set, as $\|\Psi(\gamma)\|_{OP} = 1$ for all γ.

We set G' to be the image $\Psi(G)$, and distinguish the following two cases:

Case I. G' **infinite**

First, we will use Theorem C to conclude that G' is virtually abelian (this holds trivially if G' is taken to be finite). The finite-dimensionality of $\mathcal{V}/\mathbb{C}$ (Proposition B) implies that the dimension of the vector space $\operatorname{End}(\mathcal{V}/\mathbb{C})$ is also finite. Thus, all norms in $\operatorname{End}(\mathcal{V}/\mathbb{C})$ are equivalent (meaning that its original topology and the one induced by the Lipschitz operator norm coincide) and all of its bounded sets have compact closure. Because of this, the closure $\overline{G'}$ is a compact subset of $\operatorname{End}(\mathcal{V}/\mathbb{C})$. We see that $\overline{G'}$ is a compact subgroup of $\mathrm{GL}(\mathcal{V}/\mathbb{C})$, as $\overline{G'}$ is contained completely in $\mathrm{GL}(\mathcal{V}/\mathbb{C})$, so in particular, it is a compact linear Lie group. We may apply with this setting Theorem C on G' to obtain the claim.

Now, let H' be a finite-index abelian subgroup of G', which by Lagrange's theorem H' must be also infinite. Then, H' is isomorphic to $\mathbb{Z}^k \times F$, for some $k \in \mathbb{N}$ and F finite abelian group; we denote the isomorphism by i. On the other hand, let $H \leqslant G$ be the preimage of G' under Ψ. The group H must have finite index in G (indeed, we have $[G : H] \leq [G' : H'] < \infty$, for a decomposition of G' in disjoint cosets of H' yields a decomposition of G in at most the same number of cosets of H). Moreover, if we denote by π a projection homomorphism of $\mathbb{Z}^k \times F$ onto $\mathbb{Z}$, the composite map $\pi \circ i \circ \Psi : H \longrightarrow \mathbb{Z}$ is a group epimorphism, and, therefore, by Theorem D, the group G is virtually nilpotent.

Case II. G' **finite**

Under this assumption, the trivial subgroup $\{\mathrm{id}\}$ has finite index in G'. Let us set H to be the preimage of the trivial subgroup under Ψ. The same argument as before shows that $[G : H] \leq |G'| < \infty$. By definition, the restriction of the action Ψ to H is trivial on $\mathcal{V}/\mathbb{C}$. Hence, for all $\gamma \in H$ and all $f \in \mathcal{V}$, we have

$$\gamma \cdot f + \mathbb{C} = \gamma \cdot (f + \mathbb{C}) = f + \mathbb{C},$$

that is, $\gamma \cdot f - f \in \mathbb{C}$.

Now, for every fixed $\gamma \in H$, we define a linear functional $L_\gamma \in \mathcal{V}^*$ by

$$L_\gamma f := (\gamma \cdot f - f)(e)$$

for every $f \in \mathcal{V}$. A computation (using the fact that the map $\gamma \cdot f - f$ is constant) shows that the map $L : H \longrightarrow (\mathcal{V}^*, +)$ is a group homomorphism. We remark that the dual $\mathcal{V}^*$ (together with the addition) is an infinite abelian group, and that the image $L(H)$ is a subgroup of $\mathcal{V}^*$. We consider now two subcases:

1. $L(H)$ **infinite**

   As in Case I, one concludes here that H can be mapped homomorphically onto $\mathbb{Z}$. So, again by Theorem D, G is virtually nilpotent.

2. $L(H)$ **finite**

   As before, the trivial subgroup $\{0\}$ of $\mathcal{V}^*$ has finite index in $L(H)$. Therefore, $K := L^{-1}(0)$ is a finite-index subgroup of H, and thus also of G. By definition, K acts trivially on $\mathcal{V}$: in fact, for every $\gamma \in K$
   $$L_\gamma f = (\gamma \cdot f - f)(e) = 0, \quad \text{for all } f \in \mathcal{V},$$
   but since $\gamma \cdot f - f$ is a constant, we have $\gamma \cdot f = f$.

   The K-invariance of $\mathcal{V}$ forces any Lipschitz harmonic function f to take only a finite number of values. Indeed, if $m := [G : K] < \infty$ and $G = \bigsqcup_{i=1}^m K\gamma_i$ is a decomposition of G in disjoint right cosets of K, then for any $x \in G$, there exist $k \in K$ and $i_0 \in \{1, \ldots, m\}$ such that $x = k\gamma_{i_0}$, and so
   $$f(x) = f(k\gamma_{i_0}) = \left(k^{-1} \cdot f\right)(\gamma_{i_0}) = f(\gamma_{i_0})\ .$$

   In particular, this implies that every Lipschitz harmonic function is bounded and, thus, by Liouville's theorem, is constant. This, however, contradicts Theorem A.

□

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