BASIS OF SIMULTANEOUS HECKE EIGENFORMS

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1. Introduction

The aim of this presentation is to show that there exist bases of simultaneous Hecke eigenforms (i.e., bases consisting of functions, which are eigenforms to all Hecke operators T_n) both of the space of cusp forms $S_k(\Gamma)$ and of the space of modular forms $M_k(\Gamma)$. To that end we will define a scalar product on $S_k(\Gamma)$, called the Petersson Inner Product, and show that all Hecke operators are hermitian with respect to that product. The rest will follow readily via some linear algebra. We assume familiarity with the definition of Hecke operators and Poincaré series, with their basic properties and with the corresponding notaions such as $j_\gamma(z) = (cz+d)^k$ (these have been introduced in previous talks) .

2. Construction of the Petersson Inner Product

Usually we define an inner product on spaces of functions via the use of the integral

$$< f, g >= \int_{\clubsuit} f(z)\overline{g(z)}c(z)d\xi(z)$$

for some region $\clubsuit$ on which f and g are defined, some function $c(z)$ and some measure $d\xi(z)$. Since we are trying to define the inner product on $S_k(\Gamma)$ we are dealing with functions wich are wholly defined by their values on any fundamental domain. We therefore set $\clubsuit := \Gamma\backslash H$. Now we have to choose a Γ -invariant integrand and measure. Let's first try to define the function $c(z)$ such that the integrand becomes invariant under Γ. If we compute

$$f(\gamma z)\overline{g(\gamma z)} = j_\gamma(z)f(z)\overline{j_\gamma(z)g(z)} = |j_\gamma(z)|^2 f(z)\overline{g(z)}$$

,

using $Im(\gamma z) = |j_\gamma(z)|^{-\frac{2}{k}} Im(z)$,we see that

$$f(\gamma z)\overline{g(\gamma z)}Im(\gamma z)^k = f(z)\overline{g(z)}Im(z)^k$$

,

and thus set $c(z) = c(x+iy) := y^k$.

To find the measure $d\xi(z)$, we guess that it is $d\xi(z) = d\xi(x+iy) = y^{-2}dxdy$ (It is in fact the Haar measure) . We now show that it is invariant under the action of Γ :

$$\begin{aligned} d\xi(\gamma z) &= Im(\gamma z)^{-2}\left|\frac{d(\gamma z)}{dz}\right|^2 dxdy \\ &= Im(\gamma z)^{-2}\left|\frac{det\gamma}{j_\gamma(z)^{\frac{2}{k}}}\right|^2 dxdy \\ &= Im(\gamma z)^{-2}\frac{1}{|j_\gamma(z)|^{\frac{4}{k}}}dxdy \\ &= Im(z)^{-2}dxdy \end{aligned}$$

Finally, we have to check that our potential inner product is indeed well defined. As we saw in a previous talk $f(z)y^{\frac{k}{2}}$ is bounded for any function f in the space of cusp forms. By consequence $\forall f,g \in S_k(\Gamma) \exists C \in R^+$ such that

$$\begin{aligned} \Big|\int_{\Gamma\backslash H} f(z)\overline{g(z)}y^k d\xi(z)\Big| &\le \int_{\Gamma\backslash H}\Big|f(z)\overline{g(z)}y^k\Big|d\xi(z) \\ &\le C\int_{\Gamma\backslash H} d\xi(z) = C\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{\sqrt{1-x^2}}^{\infty} y^{-2}dydx = C\frac{\pi}{3} \end{aligned}$$

The Integral is thus bounded for any two functions in the space of cusp forms and we are now free to define the

Definition (Petersson Inner Product)**.** *For any two functions f and g in $S_k(\Gamma)$ we define the Petersson Inner Product as follows:*

$$< f,g >= \int_{\Gamma\backslash H} f(z)\overline{g(z)}y^k\frac{dxdy}{y^2}$$

Notice that the Petersson Inner Product is an indeed an inner product. Recognize also that $S_k(\Gamma)$ is a finite dimensional normed vector space and therefore a Hilbert space.

3. Completeness of the Poincaré series

In this paragraph we want to see, that the Poincaré series are generators of $S_k(\Gamma)$. To that end we state the following lemma:

Lemma. *Let $f \in S_k(\Gamma)$ and P_m^k be the Poincaré series of weight k with $m \ge 1$. Then*

$$< f, P_m^k >= C_{k,m}a_m$$

where

$$C_{k,m} = \frac{\Gamma(k-1)}{(4\pi m)^{k-1}}$$

and a_m is the m-th Fourier coefficient of f.

The proof is a calculation involving the Γ - invariance of the integral defining the Petersson Inner Product, identities of the automorphy factor studied in previous talks and unfolding of the integral. For a detailed proof see Gunning's Lectures on Modular Forms. We can now prove the following

Theorem. *For $k > 2$ the set $G := \{P_m^k(z) | m \geq 1\}$ generates $S_k(\Gamma)$.*

which is what we wanted.

Proof. Let M be the subspace of $S_k(\Gamma)$ generated by G, $m \geq 1$ and $k > 2$. Then $\forall f \in M^{\perp}$:

$$< f, P_m^k >= 0$$

and using the above lemma we see that

$$C_{k,m} a_m = 0$$

Thus we see that $a_m = 0$ amounting to $f \equiv 0$, meaning that $M^{\perp} = \{0\}$ □

4. HECKE OPERATORS ARE HERMITIAN

In this section we show the following

Theorem. *Hecke Operators are hermitian with respect to the Petersson Inner Product.*

To do so we start with
Step 1: Some identities
For

$$T_n P_m(z) = n^{\frac{k}{2}} \sum_{l=1}^{\infty} c_l(n,m) e(lz)$$

$$P_m(z) = \sum_{l=1}^{\infty} c_m(l) e(lz)$$

assume the following symmetry conditions:

$$c_l(m,n) = c_l(n,m) \tag{1}$$

$$m^{1-k} c_m(l,n) = n^{1-k} c_n(l,m) \tag{2}$$

$$< T_n P_l, P_m >= C \sum_{d|(n,l)} d^{1-k} c_m \left(\frac{nl}{d^2} \right) \tag{3}$$

with $C \in R$
These may be proven by using that

$$b(l) = \sum_{d|(n,l)} d^{k-1} a \left(\frac{ln}{d^2} \right)$$

for $f = \sum_{l=1}^{\infty} a(l)e(lz)$ and $T_n f = \sum_{l=1}^{\infty} b(l)e(lz)$, wich was proven by the previous group. Sart by showing (3) using the above lemma and then proceed to prove (1) and (2) by using (3).

Step 2:

Lemma.

$$< T_n P_m, P_q >=< T_n P_q, P_m > \tag{4}$$

Proof. Recall from above that

$$< f, P_m >= C_{k,m} a_m = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} m^{1-k} a_m =: A m^{1-k} a_m$$

Using (1) and (2) we can therefore compute the following:

$$\begin{aligned}
< T_n P_m, P_q >&= A q^{1-k} a_q \\
&= A n^{\frac{k}{2}} q^{1-k} c_q(m,n) \\
&= A n^{\frac{k}{2}} n^{1-k} c_n(m,q) \\
&= A n^{\frac{k}{2}} n^{1-k} c_n(q,m) \\
&= A n^{\frac{k}{2}} m^{1-k} c_m(q.n) \\
&=< T_n P_q, P_m >
\end{aligned}$$

□

Step 3: We can finally prove the theorem

Proof. Let $f = \sum_{n=1}^{N} a_n P_n$ and$g = \sum_{m=1}^{M} a_m P_m$ be two cusp forms. First observe that $< P_n, T_k P_m >\in R$. This follows from (3) and

$$< P_a, P_b >=< T_1 P_a, P_b >=< T_1 P_b, P_a >=< P_b, P_a >$$

With this and (4) we can therefore compute the following:

$$\begin{aligned}
< T_l f, g >&=< T_l \sum_{n=1}^{N} a_n P_n, \sum_{m=1}^{M} a_m P_m > \\
&= \sum_{n=1}^{N} \sum_{m=1}^{M} a_n b_m < T_l P_n, P_m > \\
&= \sum_{n=1}^{N} \sum_{m=1}^{M} a_n b_m < T_l P_m, P_n > \\
&= \sum_{m=1}^{M} \sum_{n=1}^{N} a_n b_m < P_n, T_l P_m > \\
&=< \sum_{n=1}^{N} a_n P_n, T_l \sum_{m=1}^{M} b_m P_m > \\
&=< f, T_l g >
\end{aligned}$$

□

5. HARVEST

Theorem. *For every positive integer n there exists a basis of eigenforms of T_n for $S_k(\Gamma)$.*

This hodls since by Linear Algebra there is an orthonormal basis of eigenvectors for every hermitian liner operator on a finite dimensional vector space.

Theorem. *There is a basis of simulaneous Hecke eigenforms for $S_k(\Gamma)$.*

This also follows from a theorem of Linear Algebra, requiring the Hermitian operator to commute. This was indeed proven by the previous group.

Corollary. *There is a basis of simultaneous eigenforms for $M_k(\Gamma)$*