Bound for Fourier-coefficients and Dirichlet Series

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In this part we want to define Dirichlet series and see that they have an analytic continuation on the s-plane. More precisely, if $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ is the Fourier expansion of a function f, then we define the Dirichlet series associated with f by $L(s; f) = \sum_{n=1}^{\infty} a_n n^{-s}$. Moreover, we define the function $\Lambda_N(s; f) = (\frac{2\pi}{\sqrt{N}})^{-s}\Gamma(s)L(s; f)$ and want to show the following theorem:

Theorem 0.1 *Let $f(z)$ and $g(z)$ be holomorphic functions on H, satisfying certain conditions (see later). Let $g(z) = (-i\sqrt{N}z)^{-k} f(\frac{-1}{Nz})$ for some positive numbers k and N. Then $\Lambda_N(s; f)$ can be analytically continued to the whole s-plane, satisfying the functional equation*

$$\Lambda_N(s; f) = \Lambda_N(k - s; g).$$

1 Hecke bound for Fourier-coefficients

First we want to do a short repetition on modular forms and cusp forms.
Let $f(z)$ be a meromorphic function on the upper half-plane H, and let k be an integer. Suppose that $f(z)$ satisfies the relation

$$f(\gamma z) = (cz + d)^k f(z) \qquad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Definition 1.1 (modular form of weight k) *If $f(z)$ as defined above is holomorphic on H and at infinity (i.e. the coefficients a_n of the Fourier series are all zero for $n < 0$), then $f(z)$ is called a* modular function of weight k *for $SL(\mathbb{Z})$.*

Definition 1.2 (cusp form of weight k) *If we further have $a_0 = 0$ (i.e. the modular form vanishes at infinity), then $f(z)$ is called a* cusp form of weight k*.*

For our aim to show the theorem mentioned in the introduction, we need the following very important result about boundness of the Fourier-coefficients of a modular form:

Theorem 1.3 *If f is a cusp form of weight $2k$, then*

$$a_n = O(n^k)$$

In other words, the quotient $\frac{|a_n|}{n^k}$ remains bounded when $n \to \infty$.

Proof Because f is a cusp form, we have $a_0 = 0$ and we can factor out the factor q in the Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$, where $q = e^{2\pi i z}$. We see that

$$|f(z)| = |q||\sum_{n=1}^{\infty} a_n q^{n-1}| = O(|q|) = O(e^{-2\pi y}) \qquad (q \to 0) \tag{1}$$

with $y = Im(z)$.

We define $\phi(z) = |f(z)|y^k$. The function ϕ is continuous and formula (1) shows that ϕ tends to 0 for $y \to \infty$. This implies that ϕ is bounded. So there exists a constant M such that for all $z \in H$:

$$|f(z)| \leq M y^{-k} \tag{2}$$

Now we fix y and vary x between 0 and 1. The point $q = e^{2\pi i(x+iy)}$ runs along a circle C_y with center 0. By the residue formula, we have

$$a_n = \frac{1}{2\pi i}\int_{C_y} \tilde{f}(q)q^{-n-1}dq = \int_0^1 f(x+iy)q^{-n}dx$$

Using (2), we get from this

$$|a_n| = |\int_0^1 f(x+iy)q^{-n}dx| \leq \int_0^1 |f(x+iy)||q^{-n}|dx \leq My^{-k}\int_0^1 |q^{-n}|dx$$
$$= My^{-k}\int_0^1 |e^{-n2\pi i(x+iy)}|dx = My^{-k}e^{2\pi ny}$$

This inequality is valid for all $y > 0$. Let $y = \frac{1}{n}$, it gives $|a_n| \leq e^{2\pi}Mn^k$ and therefore $a_n = O(n^k)$. □

Lemma 1.4 *If $f = G_k$, the order of magnitude of a_n is n^{2k-1}. More precisely, there exist two constants $A, B > 0$ such that $An^{2k-1} \leq |a_n| \leq Bn^{2k-1}$.*

Proof By $\sigma_k(n)$ we denote the sum $\sum_{d|n} d^k$ of kth-powers of positive divisors of n. From Group 2 we know:
For every integer $k \geq 2$, one has

$$G_k(z) = 2\zeta(2k) + 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n=1}^{\infty}\sigma_{2k-1}(n)q^n \tag{3}$$

Formula (3) shows that there exists a constant $A > 0$ such that $a_n = (-1)^k A\sigma_{2k-1}(n)$.
Hence $|a_n| = A\sigma_{2k-1}(n) \geq An^{2k-1}$.
On the other hand we have $\frac{|a_n|}{n^{2k-1}} = A\sum_{d|n}\frac{d^{2k-1}}{n^{2k-1}} = A\sum_{d|n}\frac{1}{\frac{n}{d}^{2k-1}} \leq A\sum_{d=1}^{\infty}\frac{1}{d^{2k-1}}$
$= A\zeta(2k-1) < +\infty$. □

Theorem 1.5 *If f is a modular form of weight $2k$, but not a cusp form, then the order of magnitude of a_n is n^{2k-1}.*

Proof Write f in the form $\lambda G_k + h$ with $\lambda \neq 0$ and h a cusp form. (Group 2 showed, that this is possible.) Now we apply Lemma 1.4 on λG_k and Theorem 1.3 on h and we see that $a_n = O(n^{2k-1})$. □

2 Dirichlet Series

Definition 2.1 ($L(s; f)$) *For a holomorphic function with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$, which converges absolutely and uniformly on any compact subset of H and for which exists a $\nu > 0$ such that $f(z) = O(Im(z)^{-\nu})$ uniformly on $Re(z)$ for $Im(z) \to 0$, we put $L(s; f) = \sum_{n=1}^{\infty} a_n n^{-s}$. We call $L(s; f)$ the* Dirichlet series associated with *f.*

Then by a similar argument as in Theorem 1.3, we have $a_n = O(n^{\nu})$ and therefore $|a_n n^{-s}| = O(n^{\nu - Re(s)})$ and $L(s; f)$ converges absolutely and uniformly on any compact subset of $Re(s) > 1 + \nu$, so that it is holomorphic on $Re(s) > 1 + \nu$.

Lemma 2.2 *For a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers, put $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ for $z \in H$. If $a_n = O(n^{\nu})$ with some $\nu > 0$, then the right-hand side is convergent absolutely and uniformly on any compact subset of H, and $f(z)$ is holomorphic on H. Moreover,*

$$f(z) = O(Im(z)^{-\nu-1}) \qquad (Im(z) \to 0)$$

$$f(z) - a_0 = O(e^{-2\pi Im(z)}) \qquad (Im(z) \to \infty)$$

Proof For $\nu > 0$, we have

$$lim_{n\to\infty} \frac{n^{\nu}}{(-1)^n} \begin{pmatrix} -\nu - 1 \\ n \end{pmatrix}^{-1} = \Gamma(\nu + 1)$$

Hence there exists $L > 0$ such that

$$|a_n| \leq L(-1)^n \begin{pmatrix} -\nu - 1 \\ n \end{pmatrix}$$

for all $n \geq 0$. Put $z = x + iy$, then

$$\sum_{n=0}^{\infty} |a_n||e^{2\pi i n z}| \leq L(\sum_{n=0}^{\infty} (-1)^n \begin{pmatrix} -\nu - 1 \\ n \end{pmatrix} e^{-2\pi n y}) = L(1 - e^{-2\pi y})^{-\nu-1}$$

This implies that $f(z)$ is convergent absolutely and uniformly on any compact subset of H. Since $(1 - e^{-2\pi y}) = O(y)$ as $y \to 0$, we see that $|f(z)| = O(y^{-\nu-1})$. Moreover $f(z)$ is bounded when $y \to \infty$. Put $g(z) = \sum_{n=0}^{\infty} a_{n+1} e^{2\pi i n z}$. Since $g(z)$ also satisfies the assuption, it is bounded on a neighborhood of ∞. Therefore we obtain $f(z) - a_0 = e^{2\pi i z} g(z) = O(e^{-2\pi y})$ as $y \to \infty$. □

Definition 2.3 ($\boldsymbol{\Lambda_N(s; f)}$) *For $N > 0$ we put $\Lambda_N(s; f) = (\frac{2\pi}{\sqrt{N}})^{-s} \Gamma(s) L(s; f)$.*

Now we can prove the theorem mentioned in the introduction.

Theorem 2.4 *Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$ be holomorphic functions on H as in definition 2.1. Let $g(z)$ satisfy $g(z) = (-i\sqrt{N}z)^{-k} f(\frac{-1}{Nz})$ for some positive numbers k and N. Then $\Lambda_N(s; f)$ can be analytically continued to the whole s-plane, satisfying the functional equation*

$$\Lambda_N(s; f) = \Lambda_N(k - s; g)$$

and $\Lambda_N(s; f) + \frac{a_0}{s} + \frac{b_0}{k-s}$ is holomorphic on the whole s-plane and bounded on any vertical strip, where b_0 is the first Fourier coefficient of $g(z)$.

Proof Since there exists $\nu > 0$ such that $a_n = O(n^{\nu})$, $\sum_{n=1}^{\infty} |a_n| e^{-2\pi n t/\sqrt{N}}$ $(t > 0)$ and $\sum_{n=1}^{\infty} \int_0^{\infty} |a_n| t^{\sigma} e^{-2\pi n t/\sqrt{N}} t^{-1} dt$ $(\sigma > \nu + 1)$ are convergent. Therefore we see, for

$Re(s) > \nu + 1$

$$\begin{aligned}
\Lambda_N(s; f) &= (\frac{2\pi}{\sqrt{N}})^{-s}\Gamma(s)\sum_{n=1}^{\infty} a_n n^{-s} = \sum_{n=1}^{\infty} a_n n^{-s}(\frac{2\pi}{\sqrt{N}})^{-s}\Gamma(s) \\
&= \sum_{n=1}^{\infty} a_n(2\pi n/\sqrt{N})^{-s}\int_0^{\infty} e^{-t}t^{s-1}dt \\
&= \sum_{n=1}^{\infty}\int_0^{\infty} a_n t^s e^{-2\pi nt/\sqrt{N}}t^{-1}dt \\
&= \int_0^{\infty} t^s(\sum_{n=1}^{\infty} a_n e^{-2\pi nt/\sqrt{N}})t^{-1}dt \\
&= \int_0^{\infty} t^s(f(it/\sqrt{N}) - a_0)t^{-1}dt \\
&= \frac{-a_0}{s} + \int_1^{\infty} t^{-s}f(i/\sqrt{N}t)t^{-1}dt + \int_1^{\infty} t^s(f(it/\sqrt{N}) - a_0)t^{-1}dt
\end{aligned}$$

Since $g(z) = (-i\sqrt{N}z)^{-k}f(\frac{-1}{Nz})$, we obtain (setting $z = \frac{it}{\sqrt{N}}$)

$$\begin{aligned}
\Lambda_N(s; f) = &-\frac{a_0}{s} - \frac{b_0}{k-s} + \int_1^{\infty} t^{k-s}(g(it/\sqrt{N}) - b_0)t^{-1}dt \\
&+ \int_1^{\infty} t^s(f(it/\sqrt{N}) - a_0)t^{-1}dt
\end{aligned}$$

on $Re(s) > Max\{k, \nu + 1\}$.

By Lemma 2.2, we have, when t tends to ∞

$$\begin{aligned}
f(it) - a_0 &= O(e^{-2\pi t}), \\
g(it) - b_0 &= O(e^{-2\pi t})
\end{aligned}$$

so that $\int_1^{\infty} t^{k-s}(g(it/\sqrt{N}) - b_0)t^{-1}dt$ and $\int_1^{\infty} t^s(f(it/\sqrt{N}) - a_0)t^{-1}dt$ are absolutely convergent on any vertical strip. Therefore they are holomorphic on the whole s-plane. If we define $\Lambda_N(s; f)$ for any $s \in \mathbb{C}$, it is a meromorphic function on the whole s-plande and $\Lambda_N(s; f) + \frac{a_0}{s} + \frac{b_0}{k-s}$ is an entire function and bounded on any vertical strip.

Similarly $\Lambda_N(s; g)$ is also an entire function and bounded on the whole s-plane, and satisfies

$$\begin{aligned}
\Lambda_N(k - s; g) = &-\frac{a_0}{s} - \frac{b_0}{k-s} + \int_1^{\infty} t^{k-s}(g(it/\sqrt{N}) - b_0)t^{-1}dt \\
&+ \int_1^{\infty} t^s(f(it/\sqrt{N}) - a_0)t^{-1}dt
\end{aligned}$$

Comparing $\Lambda_N(k - s; g)$ and $\Lambda_N(s; f)$, we obtain $\Lambda_N(k - s; g) = \Lambda_N(s; f)$. □

Corollary 2.5 *If in theorem 2.4 the function f is a cusp form and if we put $f = g$ and $N = 1$, then $\Lambda_N(s; f)$ is holomorphic on the whole s-plane.*

Proof Since f is a cusp form, the coefficients $a_0 = b_0 = 0$ and holomorphicness follows. □