Hecke-Operators

Alex Maier

20th January 2007

Abstract

At the beginning we introduce the Hecke-operators and analyse some of their properties. Then we have a look at normalized eigenfunctions of them and show, that their fourier coefficients satisfy $c(n)\,c(m) = c(nm)\,, \forall\,(n,m) = 1$ and $c(p)\,c(p^n) = c(p^{n+1}) + p^{2k-1}c(p^{n-1})\,, \forall p\ prime, n \geq 1$. Next we show that the L-functions related to them can be written as Euler products. Finally we prove the properties of the τ-function.

1 Definitions

Definition 1.1 ($\mathbf{H}_n$). For $n \in \mathbb{N}$ we define:

$$\mathbf{H}_n := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad - bc = n \right\}$$

Remark 1.2. If $n = 1$, we have $\mathbf{H}_n = \Gamma$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{H}_n$$

If $f : \mathbb{H} \to \mathbb{C}$ and $k \in \mathbb{Z}$, we define the following action from M on f:

$$(f|_{2k}M)(z) := (detM)^k\,(cz+d)^{-2k}\,f(Mz)\,,\quad Mz = \frac{az+b}{cz+d}$$

$$\Rightarrow \forall \gamma_1, \gamma_2 \in M_{2\times 2}(\mathbb{Z})\,,\ det\gamma_{1,2} \neq 0 : f|_{2k}(\gamma_1\gamma_2) = (f|_{2k}\gamma_1)\,|_{2k}\gamma_2$$

So Γ acts as a group on f.

f is modular of weight $2k$ for a group $G \Leftrightarrow f|_{2k}\gamma = f,\ \forall\gamma \in G$, with G any subgroup of Γ.
From now on if we are talking about a modular form (or a cusp form or a function), the form has always weight $2k$. Similar we define $f|\gamma := f|_{2k}\gamma, \forall\gamma \in M_{2\times 2}(\mathbb{Z}),\ det\gamma \neq 0$.

2 Hecke operators

Γ acts by leftmultiplication on $\mathbf{H}_n$. So it might be interesting to consider $\Gamma\backslash\mathbf{H}_n$.

Lemma 2.1 (Set of representations).

$$\Gamma\backslash\mathbf{H}_n \cong \mathbf{M}' := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid AD = n, D > 0, 0 \leq B < D \right\} \subseteq \mathbf{H}_n$$

Proof:

- Every element in $\Gamma\backslash\mathbf{H}_n$ is representable by an element of $\mathbf{M}'$:
  Let $M \in \mathbf{H}_n, M = \pm \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Choose $c := \frac{C}{(A,C)}$, $d := \frac{-A}{(A,C)}$.
  Then (c,d) = 1 and there $\exists\ a, b \in \mathbb{Z}$ with $ad - bc = 1$ and
  $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.
  Then $cA + dC = \frac{CA}{(A,C)} - \frac{AC}{(A,C)} = 0$ and $\gamma M = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.
  $n = det(M) = det(\gamma M) = det(M') = A'D'$
  Choosing + or - we can get $D > 0$. With a suitable leftmultiplication by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ we get $0 \leq b < d$ and $M \in \mathbf{M}'$.
- Two elemens of $\Gamma\backslash\mathbf{H}_n$ are equal $\Leftrightarrow$ they are equal in $\mathbf{M}'$:
  Assume $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix}$ in $\Gamma\backslash\mathbf{H}_n$. Then an easy calculation shows that this is equivalent with $A = A', D = D'$ and $B \equiv B'$ mod D.

□

Definition 2.2 (Hecke-operator). For a modular form $f \in M_{2k}$ and $n \in \mathbb{N}$ we define an operator on the space of modular forms of weight $2k$:

$$T_n f := n^{k-1} \sum_{M \in \Gamma\backslash\mathbf{H}_n} f|_{2k} M$$

(The numerical coefficient gives formulas “without denominators”.)

Lemma 2.3. Since f is a modular form, this operator is well-defined.

Proof: Let $\{M_i\} = \mathbf{M}'$ and $\{M_i'\}$ be a different set of representations. Then $\forall i : \exists S_i \in \Gamma : M_i' = S_i M_i$. Since $f|M_i' = f|S_i M_i = f|S_i|M_i = f|M_i$ it follows directly

$$T_n f_{\{M_i'\}} = n^{k-1} \sum_i f|M_i' = n^{k-1} \sum_i f|M_i = T_n f$$

and the operater is welldefined.

□

Remark 2.4. T_n is an operator on the space of modular forms, an operator on the space of cusp forms and an operator on the space of modular functions (of weight $2k$). We prove this step by step. The first step is the next lemma:

Lemma 2.5. $T_n f$ is modular, i.e. $\forall \gamma \in \Gamma : (T_n f)\,|\gamma = T_n f$.

Proof:

$$(T_n f)\,|\gamma = n^{k-1} \sum_{M \in \Gamma\backslash \mathbf{H}_n} f|M|\gamma = n^{k-1} \sum_{M \in \Gamma\backslash \mathbf{H}_n} f|M\gamma = n^{k-1} \sum_{M' \in \Gamma\backslash \mathbf{H}_n} f|M' = T_n f$$

□

And for $M \in \mathbf{M}'$ we get:

$$(T_n f)(z) = n^{2k-1} \sum_{\substack{ad=n, d>0 \\ 0 \le b < d}} d^{-2k} f\left(\frac{az+b}{d}\right) \quad (*)$$

$$\mathcal{H} := Algebra,\ generated\ by\ the\ endomorphisms\ T_n, n = 1, 2, 3, \ldots$$

Theorem 2.6. $\mathcal{H}$ is commutative, generated by $(T_p)_{p\ prime}$.
If f is meromorphic of weight $2k$ (holomorphic on $\mathbb{H}$) $\Rightarrow T_n f$ is the same.

i)
$$T_n T_m = \sum_{d|(n,m)} d^{2k-2} T_{\frac{nm}{d^2}},\ i.e.\ T_n T_m = T_m T_n$$

ii)
$$(n, m) = 1 \Rightarrow T_n T_m = T_{nm}$$

iii)
$$p\ prime, r \in \mathbb{N} : T_p T_{p^r} = T_{p^{r+1}} + p^{2k-1} T_{p^{r-1}}$$

Proof: $(*)$ shows that $T_n f$ is meromorphic on $\mathbb{H}$ (resp. holomophic), because the sum is finite.
For i), ii) and iii), see Serre, p. 101 or Gunning, §16. □

Suppose f is a modular function, i.e. meromorphic at ∞. Then there exists a fourier expansion. Let $f(z) = \sum_{m\in\mathbb{Z}} c(m)\, q^m, q = e^{2\pi i z}$.

Proposition 2.7. $T_n f$ is a modular function, and:

$$T_n f(z) = \sum_{m\in\mathbb{Z}} \gamma(m)\, q^m \quad \Rightarrow \quad \gamma(m) = \sum_{a|(n,m), a\geq 1} a^{2k-1} c\left(\frac{mn}{a^2}\right)$$

Proof: With $M = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ and $(*)$ we have

$$(T_n f)(z) = n^{2k-1} \sum_{\substack{ad=n, d>0 \\ 0\leq b<d}} d^{-2k} f\left(\frac{az+b}{d}\right)$$

$$(T_n f)(z) = n^{2k-1} \sum_{\substack{ad=n, d>0 \\ 0\leq b<d}} d^{-2k} \sum_{m\in\mathbb{Z}} c(m)\, e^{2\pi i m M z}$$

With $\sum_{0\leq b<d} e^{2\pi i b m/d} = \begin{cases} d, & if\ d|m \\ 0, & otherwise \end{cases}$ and $m' := m/d$ we have:

$$\begin{aligned} (T_n f)(z) &= n^{2k-1} \sum_{ad=n, d>0} d^{-2k} \sum_{m\in\mathbb{Z}} \sum_{0\leq b<d} c(m)\, e^{2\pi i m(az+b)/d} \\ (T_n f)(z) &= n^{2k-1} \sum_{ad=n, d>0} \sum_{m'\in\mathbb{Z}} d^{-2k+1} c(m'd)\, q^{am'} \end{aligned}$$

Collecting powers of q and with $\mu := am', m' = \frac{\mu}{a}, d = \frac{n}{a}$ we have:

$$(T_n f)(z) = \sum_{\mu\in\mathbb{Z}} q^{\mu} \sum_{a|(n,\mu), a\geq 1} \left(\frac{n}{d}\right)^{2k-1} c\left(\frac{\mu d}{a}\right)$$

(Remark: Because of the factor n^{k-1} in the definition of the Hecke Operators we have no denominator in the following formula.)

$$\gamma(\mu) = \sum_{a|(n,\mu), a\geq 1} a^{2k-1} c\left(\frac{\mu n}{a^2}\right)$$

f is meromorphic at $\infty \Rightarrow \exists N \geq 0 : c(m) = 0, \forall m \leq -N \Rightarrow c\left(\frac{\mu d}{a}\right) = 0, \forall \mu \leq -N\frac{a}{d} \Rightarrow c\left(\frac{\mu d}{a}\right) = 0, \forall \mu \leq -Nn \Rightarrow \gamma(\mu) = 0, \forall \mu \leq -Nn \Rightarrow T_n f$ is meromorphic at ∞ and so a modular function. □

This proposition leads to 3 corollaries, which are easy to prove:

Corollary 2.8. $\gamma(0) = \sigma_{2k-1}(n)\, c(0), \gamma(1) = c(n)$

Proof sketch: Set $m = 0$, resp. $m = 1$. □

Corollary 2.9. f is a modular form (cusp form) $\Rightarrow T(n)\, f$ is a modular form (cusp form).

Proof sketch: We have $c(m) = 0, \forall m < 0$. □

Corollary 2.10. For $n = p$,prime:

- $\gamma(m) = c(pm) + p^{2k-1} c(m/p)$, if $m \equiv 0 \ (p)$,
- $\gamma(m) = c(pm)$, otherwise.

Proof sketch: Set $n = p$ and $m = lp, l \in \mathbb{N}$, resp. $m \neq lp, \forall l \in \mathbb{N}$. □

3 Eigenfunctions of the T_n's

Not all of the possible functions $f : \mathbb{H} \to \mathbb{C}$ are eigenfunctions. But in this chapter we only look at modular forms which are eigenfunctions for $T_n, \forall n \in \mathbb{N}$. We see that with the properties of the Hecke operators we can get nice properties for the fourier expansion of these functions.

Let $f = \sum_{n=0}^{\infty} c(n)\, q^n$ be a modular form of weight $2k > 0$. Assume that $\forall n \geq 1 : \exists \lambda_n \in \mathbb{C} : T_n f = \lambda_n f$.

Theorem 3.1. i) $c(1) = 0 \Rightarrow f = 0$

ii) $c(1) = 1$, i.e. f is normalized, $\Rightarrow c(n) = \lambda_n, \forall n > 1$

Proof: i) Comparison of the coefficient of q in the fourier expansion of f and $T_n f$ leads to $\lambda_n c(1) = \gamma(1) = c(n)$, by using corollary 2.8. Let us assume, that $c(1) = 0$. Then $c(n) = \lambda_n c(1) = 0, \forall n \geq 1$. So either $f = c(0)$, i.e. constant, or $c(1) \neq 0$. Since f is a modular form of weight $2k > 0$, if it is constant, $f = 0$ holds.

ii) $c(1) = 1 \Rightarrow c(n) = \lambda_n c(1) = \lambda_n$.

□

Corollary 3.2. If f and g are modular forms of weight $2k > 0$, both normalized eigenfunctions with the same eigenvalues, then they are equal, i.e.

$$c_f(1) = 1 = c_g(1) \ \textit{and}\ T_n f = \lambda_n f,\ T_n g = \lambda_n g,\ \forall n \geq 1 \Rightarrow f = g.$$

Proof: Consider $h := f - g$: $c_h(1) = 0 \overset{Thm.3.1}{\Rightarrow} h = 0$ and $f = g$ □

Corollary 3.3. i) Since for $(n, m) = 1$: $T_n T_m = T_{nm}$ we get:

$$\lambda_n \lambda_m f = T_n \lambda_m f = T_n T_m f = T_{nm} f = \lambda_{nm} f,\ \ \lambda_n \lambda_m = \lambda_{nm}$$
$$\Rightarrow c(n)\, c(m) = c(nm)\ \ \forall (n, m) = 1$$

ii) $p\ prime, n \geq 1$:

$$c(p)\, c(p^n) = c\left(p^{n+1}\right) + p^{2k-1} c\left(p^{n-1}\right)$$

□

4 Euler products

Let

$$\Phi_f(s) := \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

be the Dirichlet series defined by the $c(n)$'s in $f = \sum_{n=0}^{\infty} c(n)\, q^n$ with f a normalized Hecke eigenfunction. This series converges absolutly for $Re(s) > 2k$. (See Andrea's talk.)

Corollary 4.1 (Euler product). With $x := p^{-s}$ we have:

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - c(p)\, p^{-s} + p^{2k-1-2s}} = \prod_{p \in \mathbb{P}} \left(1 - c(p)\, x + p^{2k-1} x^2\right)^{-1}$$

Proof: The mapping $n \longmapsto c(n)$ is multiplicative for relatively prime numbers, i.e. if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \Rightarrow c(n) = c(p_1^{\alpha_1}) \cdots c(p_r^{\alpha_r})$.
$\Phi_f(s) = \prod_{p \in \mathbb{P}} \left(\sum_{n=0}^{\infty} c(p^n)\, p^{-ns}\right)$ which follows with the above property similarly as in the ζ-function. (If $c(n) = 1, \forall n > 0, \Rightarrow \Phi_f(s) = \zeta(s)$, see Simon Schieder's talk.)
With $x := p^s$ and $\Phi_{f,p}(x) = 1 - c(p)\, x + p^{2k-1} x^2$ it is enough to show:

$$\sum_{n=0}^{\infty} c(p^n)\, x^n = \frac{1}{\Phi_{f,p}(x)}, \forall p \in \mathbb{P}.$$

Or, with $\psi(x) := \left(\sum_{n=0}^{\infty} c(p^n)\, x^n\right) \Phi_{f,p}(x)$ to show $\psi(x) = 1$.
By multiplication we get $\psi(x) = 1 + \sum_{n \geq 1} a_n x^n$ for some a_n.
$a_1 = c(p) - c(p) = 0$. For $n \geq 1$:

$$a_{n+1} = c\left(p^{n+1}\right) - c(p)\, c(p^n) + p^{2k-1} c(p_{n-1}) = 0$$

by corollary 3.3. $\Rightarrow \psi(x) = 1$ which proves our claim. □

Remark 4.2. Conversely, if the Euler product holds, it implies the above multiplicativity rules of the $c(n)$'s.

5 The Δ function

Proposition 5.1. The Δ function is an eigenfunction of T_n with $T_n\Delta = \tau(n)\,\Delta$ and the normalized eigenfunction is

$$(2\pi)^{-12}\,\Delta = q\prod_{n=1}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)\,q^n$$

Proof: Since the space of cusp forms of weight 12 is of dimension 1 and stable by the T_n, Δ must be an eigenfunction. Because $\sum_{n=1}^{\infty}\tau(n)\,q^n$ is normalized we can apply theorem 3.1 and we get $\tau(n)=\lambda_n$. □

Corollary 5.2. We have

$$\tau(nm) = \tau(n)\,\tau(m)\,,\text{ if }\ (n,m)=1,$$

$$\tau(p)\,\tau(p^n) = \tau\left(p^{n+1}\right) + p^{11}\tau\left(p^{n-1}\right),\text{ if } p \text{ is a } \textit{prime},\ n\geq 1.$$

Proof: This follows immediatly from corollary 3.3. □

Remark 5.3. This properties were conjectured by Ramanujan and first proved by Mordell. But Hecke showed where these properties come from (the properties of the hecke operator).

Remark 5.4. There are similar results when the space of cusp forms of weight $2k$ has dimension 1. This happens for

$$k = 6, 8, 9, 10, 11, 13 \text{ with basis } \Delta, \Delta G_4, \Delta G_6, \Delta G_8, \Delta G_{10} \text{ and } \Delta G_{14}.$$

(For further information see Serre, p.105)