Hecke's Converse Theorem

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1 Introduction

In Andrea's Talk we saw how to get a Dirichlet series from a modular form. Moreover we saw that this Dirichlet series can be analytically continued and has a functional equation. The natural question is now: Can we go the other way? Given a Dirichlet series with a certain analytic property, can we get a corresponding modular form? - Hecke's Converse Theorem, which is the topic of this paper, states, that the answer is basically yes.

This section is a summary of what we know already about Dirichlet series and modular forms.

Definition 1.1. (Dirichlet series)
A Dirichlet series is a series of the form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad (s \in \mathbb{C})$$

for a sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers.

Theorem 1.2. (Growth of Fourier Coefficients)
If $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ is a cusp form of weight 2k, then $a_n = O(n^k)$.

Corollary 1.3. If $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ is a modular form of weight 2k but not a cusp form, then $a_n = O(n^{2k-1})$.

Definition 1.4. $(L(s,f))$
For a holomorphic function on $\mathbb{H}$ with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ we put

$$L(s; f) := \sum_{n=1}^{\infty} a_n n^{-s}.$$

We call $L(s; f)$ the Dirichlet series associated with f.

If f is a modular form, we have $a_n = O(n^{\nu})$ for a $\nu > 0$. So $L(s; f) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely and uniformly on any compact subset of $Re(s) > 1 + \nu$.

Definition 1.5. ($\Lambda(s;f)$ and $\Lambda_N(s;f)$)
For $N > 0$ we put

$$\Lambda_N(s;f) := (\frac{2\pi}{\sqrt{N}})^{-s}\Gamma(s)L(s;f).$$

Moreover we define

$$\Lambda(s;f) := \Lambda_1(s;f).$$

Theorem 1.6. Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$ and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$ be holomorphic functions on $\mathbb{H}$ satisfying $a_n = O(n^\nu)$ and $b_n = O(n^\nu)$ for a $\nu > 0$. Let k and N be positive numbers.
If $g(z) = (-i\sqrt{N}z)^{-k} f(-1/Nz)$ then both $\Lambda_N(s;f)$ and $\Lambda_N(s;g)$ can be analytically continued to the whole s-plane, satisfying the functional equation

$$\Lambda_N(s;f) = \Lambda_N(k-s;g),$$

and

$$\Lambda_N(s;f) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

So the question is, can we somehow go the converse direction in Theorem 1.6? This will be the implication of Theorem 3.1.
But first we need some analytic tools:

2 Analytic Tools

Lemma 2.1. (Stirling's estimate for $\Gamma(s)$)
$\forall s \in \mathbb{C}$ with $Arg(s) \neq \pi,\ s \neq 0:$

$$\Gamma(s) = \sqrt{2\pi} s^{s-1/2} e^{-s} e^{H(s)}$$

with

$$\lim_{|s|\to\infty} H(s) = 0$$

uniformly on any vertical strip $\nu_1 \leq \sigma \leq \nu_2$.
From that we get that $\forall\epsilon:\ 0 < \epsilon < \pi/2\ \forall z = \sigma + i\tau$ with $|\tau| > 1$:

$$|\Gamma(s)| \leq Ce^{-\frac{\pi}{2-\epsilon}|\tau|}$$

on any vertical strip $a \leq \sigma \leq b$.

Proof This is a well known estimate for $\Gamma(s)$. See for example in [3]. □

Lemma 2.2. (Inverse Mellin Transform of $\Gamma(s)$)
For $t \in \mathbb{C}$ with $Re(t) > 0$:

$$e^{-t} = \frac{1}{2\pi i}\int_{Re(s)=\sigma} \Gamma(s)t^{-s}ds \quad (\sigma > 0).$$

Proof Consider the function $F(z) := e^{-e^z} e^{\sigma z}$ as a function of $z \in \mathbb{C}$ and for a $\sigma > 0$. Obviously, F is a Schwarz function and so we can compute the Fourier transform of this function and don't have to worry about convergency:

$$\begin{aligned} \widehat{F}(x) &= \int_{-\infty}^{\infty} e^{-e^z} e^{\sigma z} e^{ixz} dz \\ &= \int_{-\infty}^{\infty} e^{-e^z} e^{(\sigma+ix)z} dz \\ &\overset{y=e^z}{=} \int_0^{\infty} e^{-y} y^{(\sigma+ix)-1} dy \\ &= \Gamma(\sigma + ix) \end{aligned}$$

So we know:

$$\check{\Gamma}(z) = e^{-e^z} e^{\sigma z} \tag{1}$$

where $\check{\Gamma}(z)$ is the inverse Fourier transform of Γ.
But we can compute $\check{\Gamma}(z)$:

$$\begin{aligned} \check{\Gamma}(z) &= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Gamma(\sigma + iy) e^{-izy} dy \\ &= \frac{1}{2\pi i} \int_{Re(s)=\sigma} \Gamma(s) e^{-zs} e^{z\sigma} ds \end{aligned}$$

With equation (1) we get $\forall z \in \mathbb{C}$:

$$e^{-e^z} e^{\sigma z} = \frac{1}{2\pi i} \int_{Re(s)=\sigma} \Gamma(s) e^{-zs} e^{z\sigma} ds$$

Since this holds $\forall z \in \mathbb{C}$, we can choose a z such that $e^z = t$ for $t \in \mathbb{C}$ with $Re(t) > 0$:

$$\begin{aligned} e^{-t} t^{\sigma} &= \int_{Re(s)=\sigma} \Gamma(s) t^{-s} t^{\sigma} ds \\ \Rightarrow e^{-t} &= \int_{Re(s)=\sigma} \Gamma(s) t^{-s} ds \end{aligned}$$

□

This is just a special case of the inverse Mellin transform. For a function $f(t)$ on $\mathbb{R}_{\geq 0}$, the Mellin transform is defined as

$$g(s) := \int_0^{\infty} f(t) t^{s-1} dt$$

if the integral is convergent.
There is an inverse transform, which, for $\sigma > 0$, has the form

$$f(t) = \frac{1}{2\pi i} \int_{Re(s)=\sigma} g(s) t^{-s} ds$$

Lemma 2.3. (Phragmen-Lindelöf-Principle)
Let $\nu_1 < \nu_2$ be two real numbers. Define

$$F := \{s \in \mathbb{C} | \nu_1 \leq Re(s) \leq \nu_2\}$$

Let ϕ be a holomorphic function on a domain containing F and satisfying

$$|\phi(s)| = O(e^{|\tau|^\delta}) \quad (|\tau| \to \infty), \quad s = \sigma + i\tau$$

uniformly on F with $\delta > 0$. Let $b \in \mathbb{R}$. If

$$|\phi(s)| = O(|\tau|^b) \quad (|\tau| \to \infty) \text{ on } Re(s) = \nu_1 \text{ and } Re(s) = \nu_2,$$

then

$$|\phi(s)| = O(|\tau|^b) \quad (|\tau| \to \infty) \text{ uniformly on } F.$$

Proof By assumption, there is $L > 0$ such that $|\phi(s)| \leq Le^{|\tau|^\delta}$.
First consider the case b=0.
Then there exists $M > 0$ such that $|\phi(s)| \leq M$ on the lines $Re(s) = \nu_1$ and $Re(s) = \nu_2$. Let m be a positive integer such that $m \equiv 2 \ mod \ 4$. Let $s = \sigma + i\tau \in \mathbb{C}$. $Re(s^m) = Re((\sigma + i\tau)^m)$ is a polynomial in σ and τ. The highest term of τ is $-\tau^m$, so we have

$$Re(s^m) = -\tau^m + O(|\tau|^{m-1}) \ \ (|\tau| \to \infty),$$

uniformly on F. Therefore $Re(s^m)$ has an upper bound on F. Now take m and N so that $m > \delta$ and $Re(s^m) \leq N \ \forall s \in F$. For all $\epsilon > 0$ we have

$$\left|\phi(s)e^{\epsilon s^m}\right| \leq Me^{\epsilon N} \text{ on } Re(s) = \nu_1 \text{ and } Re(s) = \nu_2,$$

and

$$\left|\phi(s)e^{\epsilon s^m}\right| = O(e^{|\tau|^\delta - \epsilon\tau^m}) \ \ (|\tau| \to \infty)$$

uniformly on F.
But $O(e^{|\tau|^\delta - \epsilon\tau^m}) \to 0$ uniformly on F as $|\tau| \to \infty$.
So we can use the maximum principle to see that

$$\left|\phi(s)e^{\epsilon s^m}\right| \leq Me^{\epsilon N} \quad (s \in F).$$

Now we let ϵ go to 0. So we get $|\phi(s)| \leq M$ on F, i.e. $\phi(s) = O(|\tau|^0)$.
Now let $b \neq 0$. We define a holomorphic function $\psi(s)$ by

$$\psi(s) = (s - \nu_1 + 1)^b = e^{b \log(s - \nu_1 + 1)}$$

with the principal branch of log. Since

$$Re(\log(s - \nu_1 + 1)) = \log|s - \nu_1 + 1|$$

we have

$$\psi(s) = |s - \nu_1 + 1|^b \sim |\tau|^b \quad (|\tau| \to \infty)$$

uniformly on F. Define $\phi_1(s) := \frac{\phi(s)}{\psi(s)}$. Then $\phi_1(s)$ satisfies the same assumptions as ϕ with $b = 0$. So by the case above, $\phi_1(s)$ is bounded on F. So we obtain $|\phi(s)| = O(|\tau|^b) \quad (|\tau| \to \infty)$. □

Now we have the tools we need to prove the main result on the relation between modular forms and Dirichlet series:

3 Hecke's Converse Theorem

We now want to prove, that we can go the other direction in Theorem 1.6, i.e. for every Dirichlet series with analytic continuation and the right type of functional equation, we can find a modular form.

Theorem 3.1. (Hecke, 1936)
Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$ be holomorphic functions on $\mathbb{H}$ satisfying $a_n = O(n^\nu)$ and $b_n = O(n^\nu)$ for a $\nu > 0$.
Let k and N be positive numbers.
Then the following statements are equivalent:

i) $g(z) = (-i\sqrt{N}z)^{-k} f(\frac{-1}{Nz})$.

ii) Both $\Lambda_N(s; f)$ and $\Lambda_N(s; g)$ can be analytically continued to the whole s-plane, satisfying the functional equation

$$\Lambda_N(s; f) = \Lambda_N(k - s; g),$$

and

$$\Lambda_N(s; f) + \frac{a_0}{s} + \frac{b_0}{k - s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

Proof i)⇒ ii): this is just Theorem 1.6.
ii) ⇒ i): Let f be such a function fulfilling ii). We have for $Re(y) > 0$:

$$f(iy) = \sum_{n=0}^{\infty} a_n e^{-2\pi i n y} \overset{\text{Lemma2.2}}{=} \frac{1}{2\pi i} \sum_{n=1}^{\infty} a_n \int_{Re(s)=\alpha} (2\pi n y)^{-s} \Gamma(s) ds + a_0 \quad (2)$$

for any $\alpha > 0$.
Let ν be such that $a_n = O(n^\nu)$. Choose $\alpha > \nu + 1$. So $L(s; f) := \sum_{n=1}^{\infty} a_n n^{-s}$ is uniformly convergent and bounded on Re(s)= α.
By Stirling's estimate (Lemma 2.1), $\Lambda_N(s; f) = (2\pi/\sqrt{N})^{-s}\Gamma(s)L(s; f)$ is absolutely integrable on $Re(s) = \alpha$ and therefore we can exchange the order of summation and integration:

$$\begin{aligned} f(iy) &= \frac{1}{2\pi i} \int_{Re(s)=\alpha} (2\pi y)^{-s} \Gamma(s) \sum_{n=1}^{\infty} a_n n^{-s} ds + a_0 & (3) \\ &= \frac{1}{2\pi i} \int_{Re(s)=\alpha} (\sqrt{N} y)^{-s} \Lambda_N(s; f) ds + a_0 & (4) \end{aligned}$$

$L(s; f)$ is bounded on $Re(s) = \alpha$, and so by Stirling's estimate, for any $\mu > 0$

$$|\Lambda_N(s; f)| = O(|Im(s)|^{-\mu}) \quad (|Im(s)| \to \infty) \quad (5)$$

on $Re(s) = \alpha$. Now take $\beta \in \mathbb{R}$ such that $k - \beta > \nu + 1$. With the same argument we get for any $\mu > 0$:

$$|\Lambda_N(s; g)| = |\Lambda_N(k - s; f)| = O(|Im(s)|^{-\mu}) \quad (|Im(s)| \to \infty)$$

on $Re(s) = \beta$. By assumption

$$\Lambda_N(s;f) + \frac{a_0}{s} + \frac{b_0}{k-s}$$

is bounded on the strip $\beta \leq Re(s) \leq \alpha$. So $\Lambda_N(s;f)$ has only poles at 0 and k. By Lemma 2.2, (5) holds uniformly on the domain $\beta \leq Re(s) \leq \alpha$.

Without loss of generality $\alpha > k$ and $\beta < 0$. $(\sqrt{N}y)^{-s}\Lambda_N(s;f)$ has simple poles at $s = 0$ and $s = k$, with residues $-a_0$ resp. $(\sqrt{N}y)^{-k}b_0$.
By the residue theorem and using that (5) holds uniformly on $\beta \leq Re(s) \leq \alpha$, we get

$$\begin{aligned} f(iy) &= \frac{1}{2\pi i}\int_{Re(s)=\alpha}(\sqrt{N}y)^{-s}\Lambda_N(s;f)ds + a_0 \\ &= \frac{1}{2\pi i}\int_{Re(s)=\beta}(\sqrt{N}y)^{-s}\Lambda_N(s;f)ds + (\sqrt{N}y)^{-k}b_0. \end{aligned}$$

Now we can use the functional equation for $\Lambda_N(s;f)$:

$$\begin{aligned} f(iy) &= \frac{1}{2\pi i}\int_{Re(s)=\beta}(\sqrt{N}y)^{-s}\Lambda(k-s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= \frac{1}{2\pi i}\int_{Re(s)=k-\beta}(\sqrt{N}y)^{s-k}\Lambda(s;g)ds + (\sqrt{N}y)^{-k}b_0 \\ &= (\sqrt{N}y)^{-k}(\frac{1}{2\pi i}\int_{Re(s)=k-\beta}(\sqrt{N}y)^{s}\Lambda(s;g)ds + b_0) \end{aligned}$$

Using the same calculations that led to (4), we can bring this into the form

$$f(iy) = (\sqrt{N}y)^{-k}g(\frac{-1}{iNy}).$$

f(z) and g(z) are holomorphic on $\mathbb{H}$ and so we get with $z = y/i$

$$g(z) = (\sqrt{N}z/i)^{-k}f(\frac{-1}{Nz})$$

$$g(z) = (-\sqrt{N}zi)^{-k}f(\frac{-1}{Nz})$$

and that is exactly i). □

Remark 1. Replacing $g(z)$ by $i^k g(z)$ we can reformulate Theorem 3.1:
Let $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi inz}$ and $g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi inz}$ be holomorphic functions on $\mathbb{H}$ satisfying $a_n = O(n^\nu)$ and $b_n = O(n^\nu)$ for a $\nu > 0$.
Let k and N be positive numbers.
Then the following conditions are equivalent:

i) $g(z) = (-\sqrt{N}z)^{-k}f(\frac{-1}{Nz})$.

ii) Both $\Lambda_N(s;f)$ and $\Lambda_N(s;g)$ can be analytically continued to the whole s-plane, satisfying the functional equation

$$\Lambda_N(s;f) = i^k\Lambda_N(k-s;g),$$

and

$$\Lambda_N(s;f) + \frac{a_0}{s} + \frac{i^k b_0}{k-s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

Corollary 3.2. Let $k \geq 2$ be an even number. Assume $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$ is holomorphic on $\mathbb{H}$ and $\exists \nu > 0 : \ a_n = O(n^\nu)$. Then the following statements are equivalent:

i) f(z) is a modular form of weight k

ii) $\Lambda(s;f)$ can be analytically continued to the whole s-plane, satisfying the functional equation

$$\Lambda(s;f) = (-1)^{k/2}\Lambda(k-s;f),$$

and

$$\Lambda(s;f) + \frac{a_0}{s} + \frac{(-1)^{k/2}a_0}{k-s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

Proof Set $f = g$ and $N = 1$ in Remark 1.
We have seen that for f a holomorphic function on $\mathbb{H}$ f is a modular form if and only if $f(z+1) = f(z)$ and $f(z) = z^{-k}f(-1/z)$.
Since $f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$, we have that $f(z+1) = f(z)$. So i) in Remark 1 is equivalent to f modular form of weight k.
In ii) we need that $i^k = (-1)^{k/2}$. □

There are some more general connections between modular forms and Dirichlet series / L-functions; some of them are proved recently, e.g. the Taniyama-Shimura-Weil conjecture; some of them are conjectured but still not proved, e.g. the Langlands program.

4 References

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[3] E.Freitag, R.Busam, *Funktionentheorie*, Springer-Lehrbuch