Mario Berta (Group 1)

In this section we define an action of $SL_2(\mathbb{C})$ on the Riemann sphere $\widetilde{\mathbb{C}}$. Afterwards we show that $SL_2(\mathbb{R})$ operates on the complex upper half plane H and calculate the fundamental domain of the full modular group $SL_2(\mathbb{Z})$. Furthermore we show that the group $SL_2(\mathbb{Z})/_{\pm I}$ is generated by the translation by one and the negative reciprocal map.

1 The modular group

The general linear group $GL_2(\mathbb{C})$ is defined to be the set of matrices $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b, c, d \in \mathbb{C}$ and det $g = ad - bc \neq 0$. It is easy to verify that $GL_2(\mathbb{C})$ is a group.
Let $\widetilde{\mathbb{C}}$ denote $\mathbb{C} \cup \{\infty\}$, i.e. the complex plane with a point at infinity, also known as the Riemann sphere.
For $g \in GL_2(\mathbb{C})$ and $z \in \mathbb{C}$ we define

$$gz := \frac{az+b}{cz+d} \quad g(-\frac{d}{c}) := \infty \quad g(\infty) := \frac{a}{c},\ c \neq 0 \quad g(\infty) := \infty,\ c = 0. \quad (1)$$

These maps $z \mapsto gz$ are called fractional linear transformations or Moebius transformations. One can show that these transformations map circles onto circles (where a line is considered as a special case of a circle).
The special linear group $SL_2(\mathbb{C})$ is the subgroup of $GL_2(\mathbb{C})$ consisting of matrices of determinant 1. Since a fractional linear transformations remains unchanged by multiplying all the coefficients by the same nonzero constant number, we can assume that we have $ad - bc = 1$ for all fractional linear transformations. And because $g, -g \in SL_2(\mathbb{C})$ represent the same transformation we see that the group of fractional linear transformation is isomorphic to $SL_2(\mathbb{C})/_{\pm I}$.
We now have a closer look at the special linear group over $\mathbb{R}$. Define $H := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$, i.e. H is the upper half plane.

Proposition. Every $g \in SL_2(\mathbb{R})$ preserves H.

Proof. For $g \in SL_2(\mathbb{R})$ and $z \in H$ we get:

$$\text{Im}(gz) = \text{Im}(\frac{az+b}{cz+d}) = \text{Im}(\frac{(az+b)(c\bar{z}+d)}{|cz+d|^2}) = |cz+d|^{-2}\text{Im}(z) > 0. \quad (2)$$

□

We shall now be concerned with the subgroup of $SL_2(\mathbb{R})$ consisting of matrices with integer entries. This group is called the full modular group $\Gamma := SL_2(\mathbb{Z})$. We define $\bar{\Gamma} := \Gamma/_{\pm I}$ and for any subgroup G of Γ: $\bar{G} := G/_{\pm I}$ if $-I \in G$ and $G = \bar{G}$ otherwise. Furthermore z_1, z_2 are said to be equivalent under G if $z_2 = gz_1$ for some $g \in G$. This defines an equivalence relation because the criterion to be equivalent is transitive, symmetric and reflexive. This equivalence relation divides H into a disjoint collection of equivalence classes, called orbits. We select one point from each orbit and the union of all these points is called a fundamental set of G.

2 Fundamental Domain

Let G be a subgroup of Γ. A closed subset F_G of H is called a fundamental domain of G if it has the following properties:

1. No two points in the interior of F_G are equivalent under G.
2. Every $z \in H$ is G-equivalent to a point in F_G.

Proposition. $F_\Gamma := \left\{ z \in H | -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}, |z| \geq 1 \right\}$ is a fundamental domain for Γ.

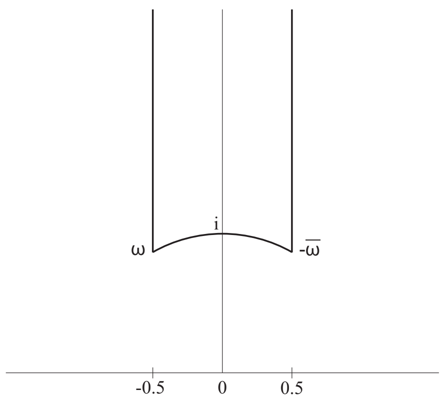

Figure 1: Fundamental Domain F_Γ, where $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$.

We first talk about the second property. Let $z \in H$ be fixed. Let Γ' be the subgroup of Γ generated by $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$: $z \mapsto -\frac{1}{z}$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$: $z \mapsto z+1$. An idea for a geometrical proof is that by applying T^j we can get that z is equivalent to a point in the strip $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. If the point lands outside the unit circle the point is in F_Γ, otherwise we can apply S to get the point outside the unit circle and then use again T^k to get the point inside the strip. One just needs to show that this is a finite process.
We now give another but precise proof.

Proof. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$ we have (by (2)): $\mathrm{Im}(z) = |cz+d|^{-2}\mathrm{Im}(z)$. Since $c, d \in \mathbb{Z}$ and $ad - bc = 1$, the numbers $|cz+d|$ are bounded away from zero. So there is some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma'$ such that $\mathrm{Im}(\gamma z)$ is maximal (i.e. $|cz+d|$ minimal). Replacing γ by $T^j\gamma$ wlog we can assume that γz is in the strip

$-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$. If $|\gamma z| \geq 1$ the proof of the second property is finished. Otherwise we have $\mathrm{Im}(S\gamma z) = (\frac{1}{\gamma z+0})^2 \mathrm{Im}(\gamma z) = \mathrm{Im}(\frac{\gamma z}{|\gamma z|^2}) > \mathrm{Im}(\gamma z)$, which contradicts our choice of $\gamma \in \Gamma'$ such that $\mathrm{Im}(\gamma z)$ is maximal.
We now proof the first property. Suppose that $z_1, z_2 \in F_\Gamma$ are Γ-equivalent. Wlog we assume $\mathrm{Im}(z_2) \geq \mathrm{Im}(z_1)$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ be such that $z_2 = \gamma z_1$. By (2) we get $|cz_1 + d| \leq 1$. Since z_1 is in F_Γ, $d \in \mathbb{Z}$ and $\mathrm{Im}(z_1) \geq \frac{\sqrt{3}}{2}$, this inequality does not hold for $|c| \geq 2$. This leaves the four cases (i) $c = 0, d = \pm 1$, (ii) $c = \pm 1, d = 0$, (iii) $c = d = 1$ and (iv) $c = \mp d = \pm 1$. For (i) we get $\gamma = \pm\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right) = \pm T^b$. For $b = 0$ we have $\gamma = \pm I$ and $z_1 = z_2$. $|b| > 1$ is obviously not possible and for $b = \pm 1$ we get $\gamma = \pm\left(\begin{smallmatrix} 1 & \pm 1 \\ 0 & 1 \end{smallmatrix}\right)$ what implies that z_1, z_2 are on the vertical lines $\mathrm{Re}(z) = \pm\frac{1}{2}$. In case (ii) it is easy to see that $\gamma = \pm\left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right) = \pm T^a S$. So z_1, z_2 have to be on the unit circle (otherwise $|1z + 0| \leq 1$ is not fulfilled). For $a = 0$, z_1, z_2 are symmetrically located on the unit circle with respect to the imaginary axis. $|a| > 1$ is not possible and for $a = \pm 1$ we can see easily that $\gamma = \pm T^{\pm 1} S$ and $z_1 = z_2 = \pm\frac{1}{2} + \frac{\sqrt{-3}}{2}$. The third case gives $\gamma = \pm\left(\begin{smallmatrix} a & a-1 \\ 1 & 1 \end{smallmatrix}\right) = \pm T^a \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $z_1 = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$. $a = 0$ gives us $\gamma = \pm\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ what corresponds to the transformation $z \mapsto \pm\frac{1}{z+1}$ and hence we get $z_2 = STz_1 = z_1$ for z_1 as above. For $a = 1$ we have $\gamma = \pm T\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right) = \pm TST$ and therefore $z_2 = z_1 + 1$. Again $|a| > 1$ is not possible and it is easy to check that $a = -1$ is not possible either. (iv) is handled in the same manner as (iii). So in no case z_1 and z_2 belong to the interior of F_Γ unless $\pm\gamma$ is the identity and $z_1 = z_2$. □

In this proof we established two other remarkable facts.

Proposition. Two distinct points z_1, z_2 on the boundary of F_Γ are Γ-equivalent only if $\mathrm{Re}(z_1) = \pm\frac{1}{2}$ and $z_2 = z_1 + 1$ or if z_1 is on the unit circle and $z_2 = -\frac{1}{z_1}$.

Proof. See above. □

In the next proposition we need the following notation. $G_z := \{z \in G | gz = z\}$ is the isotropy subgroup of z under the action of G.

Proposition. If $z \in F_\Gamma$ then $\Gamma_z = \pm I$ except for:

- $\Gamma_{z=i} = \pm\{I, S\}$
- $\Gamma_\omega = \pm\left\{I, ST, (ST)^2\right\}$ for $\omega = -\frac{1}{2} + \frac{\sqrt{-3}}{2}$
- $\Gamma_{-\bar{\omega}} = \pm\left\{I, TS, (TS)^2\right\}$

Proof. Notice that $(ST)^3 = (TS)^3 = I$ and see the proof above. □

Points with nontrivial isotropy subgroup are called elliptic points. We can get another useful proposition by the proof above.

Proposition. The group $\bar{\Gamma}$ is generated by the two elements S and T.

Proof. Let Γ' be the subgroup of Γ generated by S and T. Let z be any point in the interior of F_Γ. Let g be an element of Γ. Consider $gz \in H$. We know that there exists $\gamma \in \Gamma'$ such that $\gamma(gz) = \gamma gz \in F_\Gamma$. But since z is in the interior of F_Γ we have $\gamma g = \pm I$. And because γ has an inverse we get $g = \pm\gamma^{-1} \in \Gamma'$. So any $g \in \Gamma$ is up to a sign in Γ'. $\square$

We can identify all Γ-equivalent points on the boundary of F_Γ and call this set F_Γ^g. F_Γ^g is in one-to-one correspondence to the set of Γ-equivalence classes in H, which we denote Γ_H.

One can show that the fundamental domain $\bar{F}_\Gamma^g$ of $\bar{H} := H \cup \mathbb{Q} \cup \{\infty\}$ is homeomorphic to S^2.