Pro memoria

$$E_k(z) = 1 + C_k \sum_{m=1}^{\infty} \sigma_{k-1}(m) q^m, \quad q = e^{2\pi i z}$$

with

k	2	4	6	8	10	12
C_k	-24	240	-504	480	-264	$\frac{65520}{691}$

Exercise 1

a.) It is clear that $E_4^2 \in \mathbb{M}_8$ and $E_4 E_6 \in \mathbb{M}_{10}$.
Since $\dim \mathbb{M}_8 = \dim \mathbb{M}_{10} = 1$, we get $E_4^2 = cE_8$ and $E_4E_6 = dE_{10}$. Comparing the coefficient of q^0 we see $c = d = 1$

b.) We have

$$\begin{aligned} E_8 &= E_4^2 = \left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^m\right)^2 = 1 + 480 \sum_{m=1}^{\infty} \sigma_3(m) q^m + \left(240 \sum_{m=1}^{\infty} \sigma_3(m) q^m\right)^2 \\ &= 1 + 480 \sum_{m=1}^{\infty} \sigma_3(m) q^m + 240^2 \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \sigma_3(n)\sigma_3(m-n) q^m \end{aligned}$$

Comparing the coefficient of q^m, $m \geq 1$ we get

$$480\sigma_7(m) = 480\sigma_3(m) + 240^2 \sum_{n=1}^{m-1} \sigma_3(n)\sigma_3(m-n)$$

Finally we divide by 480 and get

$$\sigma_7(m) = \sigma_3(m) + 120 \sum_{n=1}^{m-1} \sigma_3(n)\sigma_3(m-n)$$

Analog

$$E_{10} = E_4E_6 = 1 + \sum_{m=1}^{\infty} \left(240\sigma_3(m) - 504\sigma_5(m)\right) q^m - 240 \cdot 504 \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \sigma_3(n)\sigma_5(m-n) q^m$$

and

$$11\sigma_9(m) = 21\sigma_5(m) - 10\sigma_3(m) + 5040 \sum_{n=1}^{m-1} \sigma_3(n)\sigma_5(m-n)$$

Exercise 2

Let $k \geq 2$. It is clear that $g(z)$ is holomorphic in $\mathbb{H}$ and periodic with Period 1. Watching at the Fourier expansion of each factor, one also sees immediately that $g(z)$ is holomorphic at infinity. So we only have to show $g(-1/z) = z^{k+2} g(z)$. We have

$$E_2(-1/z) = z^2 E_2(z) + \frac{12z}{2\pi i}$$

We differentiate $f(-1/z) = z^k f(z)$ and multiply by z^2 and get

$$f'(-1/z) = kz^{k+1}f(z) + z^{k+2}f'(z)$$

$$\begin{aligned}
g(-1/z) &= \frac{1}{2\pi i}f'(-1/z) - \frac{k}{12}E_2(-1/z)f(-1/z) \\
&= \frac{1}{2\pi i}\left(kz^{k+1}f(z) + z^{k+2}f'(z)\right) - \frac{k}{12}\left(z^2E_2(z) + \frac{12z}{2\pi i}\right)z^k f(z) \\
&= z^{k+2}\left(\frac{1}{2\pi i}f'(z) - \frac{k}{12}E_2(z)f(z)\right) + \left(\frac{1}{2\pi i}kz^{k+1}f(z) - \frac{k}{12}\frac{12z}{2\pi i}z^k f(z)\right) \\
&= z^{k+2}g(z)
\end{aligned}$$

This implies $g(z) \in \mathbb{M}_{k+2}$.
Now will calculate the Fourier coefficient of q^0 of $g(z)$. We have $f(z) = a_0 + a_1q + O(q^2)$ and $E_2(z) = 1 - 24q + O(q^2)$. Notice $(q^m)' = 2\pi i m q^m$

$$g(z) = a_1q + O(q^2) - \frac{k}{12}(1 - 24q + O(q^2))(a_0 + a_1q + O(q^2)) = \frac{-k}{12}a_0 + O(q)$$

We get immediately $g(z)$ is a cusp form if and only if $f(z)$ is a cusp form.
<u>Exercise 3</u>
a.) Using Exercise 2 with $f = -3E_4$ we see $E_4E_2 - \frac{3}{2\pi i}E_4'$ is a modular form with coefficient of q^0 equal to 1. Since $\dim \mathbb{M}_6 = 1$ we get $E_6 = E_4E_2 - \frac{3}{2\pi i}E_4'$
b.) We get

$$\begin{aligned}
E_6 &= E_4E_2 - \frac{3}{2\pi i}E_4' \\
&= \left(1 + 240\sum_{m=1}^{\infty}\sigma_3(m)q^m\right)\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^m\right) - \frac{3}{2\pi i}\left(-24\sum_{m=1}^{\infty}\sigma_1(m)(2\pi i)mq^m\right) \\
&= 1 + \sum_{m=1}^{\infty}\left(240\sigma_3(m) + 24(3m-1)\sigma_1(m)\right)q^m - 24\cdot 240\sum_{m=1}^{\infty}\sum_{n=1}^{m-1}\sigma_1(n)\sigma_3(m-n)q^m
\end{aligned}$$

It follows for $m \geq 1$

$$-504\sigma_5(m) = 240\sigma_3(m) + 24(3m-1)\sigma_1(m) - 24\cdot 240\sum_{n=1}^{m-1}\sigma_1(n)\sigma_3(m-n)$$

Dividing by -24

$$21\sigma_5(m) = (1-3m)\sigma_1(m) - 10\sigma_3(m) + 240\sum_{n=1}^{m-1}\sigma_1(n)\sigma_3(m-n)$$

<u>Exercise 4</u>

a.)It is clear $E_{12} - E_6^2 \in \mathbb{M}_{12}$. Now we take a look at the Fourier expansion:

$$E_{12} - E_6^2 = 1 + \frac{65520}{691} q + O(q^2) - \left(1 - 504q + 0(q^2)\right)^2 = \left(\frac{65520}{691} + 1008\right) q + O(q^2)$$

Since $a_0 = 0$ we see that $E_{12} - E_6^2$ is a cusp form. But $\dim \mathbb{S}_{12} = 1$.
So we get $E_{12} - E_6^2 = c\Delta$. On the other Hand $\Delta = (2\pi)^{12} \sum_{m=1}^{\infty} \tau(m)$ with $\tau(1) = 1$ and $\tau_m \in \mathbb{Z}\ \forall m$. So $c = (2\pi)^{-12} \frac{762048}{691}$

b.) For simplicity we write $E_6^2(z) = 1 + \sum_{m=1}^{\infty} b_m q^m$ with $b_m \in \mathbb{Z}$. Therefore

$$\frac{762048}{691} \sum_{m=1}^{\infty} \tau(m) = \frac{65520}{691} \sum_{m=1}^{\infty} \sigma_{11}(m) q^m - \sum_{m=1}^{\infty} b_m$$

We multiply with 691 and get

$$(65520 + 1008 \cdot 691)\tau(m) - 65520\sigma_{11}(m) = -691 b_m$$

and finally

$$65520(\tau(m) - \sigma_{11}(m)) = 691 \underbrace{(-b_m - 1008)}_{\in \mathbb{Z}}$$

But 691 is a prime and $65520 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ is not divisible by 691. Therefore $\tau(m) - \sigma_{11}(m)$ is divisible by 691.

<u>Exercise 5</u>

Remark: I will use $\widehat{f}(y) = \int f(x) e^{-2\pi i x y} dx$, as in Koblitz. It is for this reason why the factor $\chi(-1)$ in a.) does not appear.

a.) The Idea is to use the Poisson Summation formula and the periodicity of χ. We define therefore $f_n(x) := f(Nx + n),\ 0 \le n \le N-1$. It follows

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \chi(n) f(n) &= \sum_{m=0}^{N-1} \sum_{n=-\infty}^{\infty} \chi(nN+m) f(nN+m) = \sum_{m=0}^{N-1} \left(\chi(m) \sum_{n=-\infty}^{\infty} \right) f_m(n) \\ &= \sum_{m=0}^{N-1} \left(\chi(m) \sum_{n=-\infty}^{\infty} \widehat{f_m}(n) \right) \end{aligned}$$

Notice that the first sum is absolutely convergent since $x^2 f(x) \le 1$ for x big enough and so we are allowed to change the oder of summation. Now we use $\widehat{f_m}(n) = \widehat{f}(m + nN) = \frac{1}{N} e^{2\pi i nm/N} \widehat{f}(\frac{n}{N})$. This can be seen for example by direct calculation. We get

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \chi(n) f(n) &= \sum_{m=0}^{N-1} \left(\chi(m) \sum_{n=-\infty}^{\infty} \frac{1}{N} e^{2\pi i nm/N} \widehat{f}\left(\frac{n}{N}\right) \right) = \frac{1}{N} \sum_{n=-\infty}^{\infty} \left(\widehat{f}\left(\frac{n}{N}\right) \sum_{m=0}^{N-1} \chi(m) e^{2\pi i nm/N} \right) \\ &=^{(i)} \frac{1}{N} \sum_{n=-\infty}^{\infty} \widehat{f}\left(\frac{n}{N}\right) \overline{\chi(n)} \tau(\chi) = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \widehat{f}\left(\frac{n}{N}\right) \end{aligned}$$

b.) We set $f(x) := e^{-\pi t x^2}$. Then $\widehat{f}(y) = \sqrt{\frac{1}{t}} e^{-\pi y^2/t}$. If we apply this to a.) we get

$$2\Theta_\chi(t) = \sum_{n=-\infty}^{\infty} \chi(n) e^{-\pi t n^2} = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \sqrt{\frac{1}{t}} e^{-\pi (\frac{n}{N})^2/t} = 2 \frac{\tau(\chi)}{N\sqrt{t}} \Theta_{\overline{\chi}} \left(\frac{1}{tN^2} \right)$$

c.) Notice $\chi(-n) e^{-\pi(-n)^2 t} = \chi(-1)\chi(n) e^{-\pi n^2 t}$. Therefore we have to use an other Θ for $\chi(-1) = -1$

We will use again a.), but with $f(x) = x e^{-\pi t x^2}$. Therefore we need to calculate the fourier transformation of f. Let $T > 0$ be arbitrary. Then

$$\int_T^T \underbrace{s e^{-\pi t s^2}}_{u'} \underbrace{e^{-2\pi i y s}}_{v} ds = \left[\frac{-1}{2\pi t} e^{-\pi t s^2} e^{-2\pi i y s} \right]_{s=-T}^{T} - \int_T^T \frac{-1}{2\pi t} e^{-\pi t s^2} (-2\pi i y) e^{-2\pi i y s}$$

We get

$$\widehat{f}(y) = \lim_{T \to \infty} \int_{-T}^{T} s e^{-\pi t s^2} e^{-2\pi i y s} ds = \frac{-iy}{t} \int_{-\infty}^{\infty} e^{-\pi t s^2} e^{-2\pi i y s} ds = \frac{-iy}{t} \sqrt{\frac{1}{t}} e^{-\pi y^2/t} = \frac{-iy}{t^{3/2}} e^{-\pi y^2/t}$$

As in b.)

$$\begin{aligned} 2\Theta_\chi(t) &= \sum_{n=-\infty}^{\infty} n\chi(n) e^{-\pi t n^2} = \frac{\tau(\chi)}{N} \sum_{n=-\infty}^{\infty} \overline{\chi(n)} \frac{-in}{N t^{3/2}} e^{-\pi(\frac{n}{N})^2/t} = \frac{-i\tau(\chi)}{N^2 t^{3/2}} \sum_{n=-\infty}^{\infty} n \overline{\chi(n)} e^{-\pi(\frac{n}{N})^2/t} \\ &= 2 \frac{-i\tau(\chi)}{N^2 t^{3/2}} \Theta_{\overline{\chi}}(t) \end{aligned}$$

d.) As usual we write $s = \sigma + it$. For shortness we write $\Theta = \Theta_\chi$ and $\Theta_c = \Theta_{\overline{\chi}}$
<u>Claim</u> $f(s) = \int_0^\infty \Theta(x) x^{s-1} dx$ is an entire function
<u>Proof</u> Let $K \subset \mathbb{C}$ be compact. We have

$$\int_0^\infty |\Theta(x) x^{s-1}| dx = \underbrace{\int_0^1 \Theta(x) x^{\sigma-1} dx}_{(1)} + \underbrace{\int_1^\infty \Theta(x) x^{\sigma-1} dx}_{(2)}$$

Existence of (2): We set $T := \max\{2, \sup_{s \in K} \sigma\}$
Notice $\chi(0) = 0$ and $|\chi(n)| = 1$

$$\begin{aligned} \int_1^\infty \Theta(x) x^{\sigma-1} dx &\le \int_1^\infty \Theta(x) x^{T-1} dx \le \int_1^\infty \sum_{n=1}^{\infty} n e^{-\pi n^2 x} x^{T-1} dx \\ &\le \sum_{n=1}^{\infty} \int_{1/\pi n^2}^{\infty} n e^{-y} \left(\frac{y}{\pi n^2} \right)^{T-1} \frac{dy}{\pi n^2} \le \pi^{-T} \sum_{n=1}^{\infty} \frac{1}{n^{2T-1}} \int_0^\infty e^{-\pi y} y^{T-1} dy \\ &= \pi^{-T} \Gamma(T) \zeta(2T-1) \end{aligned}$$

Existence of (1):We set $y = \frac{1}{N^2 x}$ and get

$$\int_0^1 \Theta(x)x^{\sigma-1}dx = \int_{1/N^2}^{\infty} \Theta\left(\frac{1}{N^2 y}\right)\frac{1}{y^{\sigma+1}}\frac{1}{N^\sigma}dy = C(\epsilon)\int_{1/N^2}^{\infty} \Theta_c(y)y^{1/2+\epsilon}\frac{1}{y^{\sigma+1}}\frac{1}{N^\sigma}dy$$

Now we can prossed as for (2) □

<u>Claim</u> $\Lambda(2s-\epsilon,\chi) = \int_0^\infty \Theta(x)x^{s-1}dx$ for σ big enough.

<u>Proof</u> Case 1: $\chi(-1) = 1$. We get

$$\begin{aligned}\int_0^\infty \Theta(x)x^{s-1}dx &= \int_0^\infty \sum_{n=1}^\infty \chi(n)e^{-\pi n^2 x}x^{s-1}dx = \sum_{n=1}^\infty \chi(n)\int_0^\infty e^{-\pi n^2 x}x^{s-1}dx \\ &= \sum_{n=1}^\infty \chi(n)\int_0^\infty e^{-y}\left(\frac{y}{\pi n^2}\right)^{s-1}\frac{dy}{\pi n^2} = \pi^{-s}\sum_{n=1}^\infty \frac{\chi(n)}{n^{2s}}\Gamma(s) \\ &= \pi^{-s}\Gamma(s)L(2s,\chi) = \Lambda(2s,\chi)\end{aligned}$$

Case 2: $\chi(-1) = -1$. We use $-n\chi(-n) = n\chi(n)$ and get

$$\begin{aligned}\int_0^\infty \Theta(x)x^{s-1}dx &= \int_0^\infty \sum_{n=1}^\infty n\chi(n)e^{-\pi n^2 x}x^{s-1}dx = \sum_{n=1}^\infty n\chi(n)\int_0^\infty e^{-\pi n^2 x}x^{s-1}dx \\ &= \sum_{n=1}^\infty n\chi(n)\int_0^\infty e^{-y}\left(\frac{y}{\pi n^2}\right)^{s-1}\frac{dy}{\pi n^2} = \pi^{-s}\sum_{n=1}^\infty \frac{\chi(n)}{n^{2s-1}}\Gamma(s) \\ &= \pi^{-s}\Gamma(s)L(2s-1,\chi) = \Lambda(2s-1,\chi)\end{aligned}$$

□

<u>Functional Equation</u>

$$\begin{aligned}\frac{(-i)^\epsilon\tau(\chi)}{N^{1+\epsilon}}\Lambda(2s-\epsilon,\overline{\chi}) &= \frac{(-i)^\epsilon\tau(\chi)}{N^{1+\epsilon}}\int_0^\infty \Theta_c(x)x^{s-1}dx = \frac{(-i)^\epsilon\tau(\overline{\chi})}{N^{1+\epsilon}}\int_0^\infty \Theta_c\left(\frac{1}{N^2 y}\right)\frac{1}{N^{2s}}\frac{dy}{y^{s+1}} \\ &= \frac{1}{N^{2s}}\int_0^\infty \Theta(y)y^{(1/2+\epsilon-s)-1}ds = \frac{1}{N^{2s}}\Lambda(1+\epsilon-2s,\chi)\end{aligned}$$

Now we set $s = \frac{-s+1+\epsilon}{2}$ and get

$$\Lambda(s,\chi) = (-i)^\epsilon\tau(\chi)\frac{N^{-s+1+\epsilon}}{N^{1+\epsilon}}\Lambda(1-s,\overline{\chi}) = (-i)^\epsilon\tau(\chi)N^{-s}\Lambda(1-s,\overline{\chi})$$