THE POINCARÉ SERIES

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In this lecture we will discuss a particular method to construct modular forms. This is actually an elusively rare thing: only two such constructions are known to us, the first one we will come to in a minute and next week we will see the second one.

In the last lecture we saw that the modular discriminant is a simultaneous Hecke eigenform, and, hence, has an Euler product expansion. This revealed Δ's underlying arithmetic properties.

Our construction of modular forms will lead us to the Poincaré series, and with the help of the Petersson inner product we will see in today's second lecture that actually more is true: the space of modular forms even has a basis consisting solely of simultaneous Hecke eigenforms.

Construction of Modular Forms

Let us first recall the definition of a modular form. It is a function $f : \mathbf{H} \to \mathbf{C}$ such that

- *(Holomorphy)* f is holomorphic on $\mathbf{H}$ and at $i\infty$,
- *(Automorphy)* $f(\gamma z) = (cz+d)^k f(z) \quad \forall \gamma \in \Gamma$.

For the purpose of constructing such functions we begin with constructing functions that satisfy the automorphy condition, but not yet the holomorphy condition.

The automorphy condition, on the other hand, amounts to saying that we are interested in functions that transform under Γ according to a given factor. To prevent, however, the specific factor to interfere with any ideas, we push it out of our discussion.

Definition (Automorphy factor)**.** *A function $j : \Gamma \times \mathbf{H} \to \mathbf{C}^\times$ is called automorphy factor, if for each $\alpha, \beta, \gamma \in \Gamma$ j_γ is holomorphic on $\mathbf{H}$ and $j_{\alpha\beta}(z) = j_\alpha(\beta z) j_\beta(z)$.*

We immediately generalize and replace the automorphy condition by

- *(Automorphy)* $f(\gamma z) = j_\gamma(z) f(z) \quad \forall \gamma \in \Gamma$

Consider the simplest case for j, i.e., $j \equiv 1$. The automorphy condition now becomes the invariance condition $f(\gamma z) = f(z)$, but those functions can be written as an average over Γ: $f(z) = \sum_{\alpha \in \Gamma} h(\alpha z)$ for some h.

Our strategy is to adapt the idea of averaging over Γ to arbitrary automorphy factors j. Hence, we replace $h(\alpha z)$ in the sum by the more generic term $h_\alpha(z)$ for some function h.

We formally define $f(z) := \sum_{\alpha \in \Gamma} h_\alpha(z)$, but not all such f satisfy the automorphy condition yet and we investigate what the condition means for functions defined in this way.

$$\textit{(Automorphy)} \quad \Longleftrightarrow \quad \sum_{\alpha \in \Gamma} h_\alpha(\gamma z) = \sum_{\alpha \in \Gamma} j_\gamma(z) h_\alpha(z)$$
$$= \sum_{\alpha \in \Gamma} j_\gamma(z) h_{\alpha\gamma}(z) \quad \forall \gamma \in \Gamma.$$

The equation is simplified if we not only ask the sums to be equal, but even ask for pointwise equality,

$$h_\alpha(\gamma z) = j_\gamma(z) h_{\alpha\gamma}(z) \quad \forall \alpha, \gamma \in \Gamma.$$

The case $\alpha = 1$ reveals that $h_\alpha(z)$ must now take on a particular form,

$$h_1(\gamma z) = j_\gamma(z) h_\gamma(z) \quad \Longleftrightarrow \quad h_\gamma(z) = \frac{h_1(\gamma z)}{j_\gamma(z)} \quad \forall \gamma \in \Gamma.$$

With this identity the automorphy condition follows from

$$\frac{h_1(\alpha\gamma z)}{j_\alpha(\gamma z)} = j_\alpha(z) \frac{h_1(\alpha\gamma z)}{j_{\alpha\gamma}(z)} \quad \Longleftrightarrow \quad j_{\alpha\gamma}(z) = j_\alpha(\gamma z) j_\alpha(z) \quad \forall \alpha, \gamma \in \Gamma$$

Automorphy factors evidently satisfy the last expression, and—featuring the abbreviation $h := h_1$—our construction becomes

$$f(z) := \sum_{\alpha \in \Gamma} \frac{h(\alpha z)}{j_\alpha(z)}.$$

If the sum converges absolutely, then f satisfies the automorphy condition.

Unfortunately, it turns out that searching for the right function h is still unfeasible: the main reason is that there may be infinitely many γ with $j_\gamma \equiv 1$, e.g., in our case for each $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$.

A quick examination, however, reveals that these redundant terms are a subgroup of Γ,

$$\Gamma_\infty := \{\gamma \in \Gamma \mid j_\gamma \equiv 1\},$$

and the idea to mod Γ_∞ away raises hopes.

To actually do this, we assume the function h invariant under Γ_∞ and we merely need to show that $\alpha \mapsto \frac{h(\alpha z)}{j_\alpha(z)}$ only depends on the cosets mod Γ_∞. Let $\alpha = \beta\alpha'$ with $\beta \in \Gamma_\infty$ and notice that $j_\alpha(z) = j_{\beta\alpha'}(z) = j_\beta(\alpha' z) j_{\alpha'}(z) = j_{\alpha'}(z)$, and $h(\alpha z) = h(\beta\alpha' z) = h(\alpha' z)$.

We finally derived a beautiful formula to construct functions that satisfy the automorphy condition if h is Γ_∞-invariant and the sum converges absolutely,

$$f(z) := \sum_{\alpha \in \Gamma_\infty \backslash \Gamma} \frac{h(\alpha z)}{j_\alpha(z)} = \sum_{\alpha \in \Gamma_\infty \backslash \Gamma} (h|\alpha)(z).$$

If the series satisfies the holomorphy condition, too, we solved our problem and found a modular form.

Construction of the Poincaré Series

We are ready to transfer the general construction to our setting with $\Gamma = SL_2(\mathbf{Z})$ and $j_\gamma(z) = (cz + d)^k$.

Note that the matrices in Γ_∞ are those with $\pm(0, 1)$ in the second row. Because they have determinant 1, we have $\Gamma_\infty = \{\pm T^n \mid n \in \mathbf{Z}\}$. Hence, Γ_∞-invariant functions are those with period 1. The simplest such function is $e(mz) := e^{2\pi i m z}$, which is our choice for h.

To figure out the cosets in $\Gamma_\infty \backslash \Gamma$, note that

$$M = \pm T^n M' \quad \Longleftrightarrow \quad (c, d) = \pm(c', d').$$

$$M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma \quad \Longrightarrow \quad \gcd(c, d) = 1,$$

$$\gcd(c, d) = 1 \quad \Longrightarrow \quad \exists M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma.$$

It follows that

$$\Gamma_\infty \backslash \Gamma \cong \{(c,d) \mid c \geq 0,\ \gcd(c,d) = 1\}.$$

With the understanding that γ always denotes a matrix in Γ with c and d in the second row, we arrive at the Poincaré series.

Definition (Poincaré series)**.**

$$P_m^k(z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{e(m\gamma z)}{j_\gamma(z)} = \sum_{\substack{c,d \in \mathbf{Z} \\ c \geq 0 \\ \gcd(c,d)=1}} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k}$$

is called the m^{th} *Poincaré series of weight* k*.*

Theorem. *If* $k, m \in \mathbf{N}$*,* $m \geq 0$*, and* $k > 2$*, then the Poincaré series* P_m^k *is a modular form of weight* k*.*

Proof. As per construction, the automorphy condition is satisfied if the series converges absolutely uniformly on compact subsets of $\mathbf{H}$. The remainder of the proof is done similarly as in the proof of modularity of the Eisenstein series. □

Fourier Expansion of the Poincaré Series

The Fourier expansion of P_m^k will use Kloosterman sums.

Definition (Kloosterman sum)**.** *The sum*

$$k(m,n,c) := \sum_{\substack{d \bmod c \\ \gcd(c,d)=1}} e\left(\frac{m\bar{d} + nd}{c}\right) \qquad \text{with } d\bar{d} \equiv 1 \pmod{c}$$

is called Kloosterman sum.

Theorem. *Let* a_n *be the* n^{th} *Fourier coefficient of* P_m^k*.*
If $m = 0$*, then*

$$a_n = \frac{(2\pi i)^k \sigma_{k-1}(n)}{(k-1)!\zeta(k)};$$

if $m \geq 1$*, then*

$$a_n = \sum_{c>0} k(m,n,c) \frac{(2\pi i)^k}{c} \left(\frac{n}{m}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

where J_n *is the* n^{th} *Bessel function.*

Proof. The Fourier coefficient a_n is given by

$$a_n = \int_{0+iy}^{1+iy} P_m^k(z) e(-nz) dz = \sum_{\substack{c,d \in \mathbf{Z} \\ c \geq 0 \\ \gcd(c,d)=1}} \int_{0+iy}^{1+iy} \frac{e(m\gamma z)}{j_\gamma(z)} e(-nz) dz$$

The sum is complicated, but note that for all $c > 0$ and $d \in \mathbf{Z}$ with $\gcd(c,d) = 1$ there exists $l \in \mathbf{Z}$ and $d' \bmod c$ with $\gcd(c,d') = 1$ such that $d = lc + d'$. This formula generates exactly all d that are relatively prime to c. We split off the terms with $c = 0$ and rewrite the remaining sum:

$$a_n = \sum_{c=0, d=\pm 1} \int_{0+iy}^{1+iy} e\left((m-n)z\right) dz + \sum_{\substack{c>0 \\ d' \bmod c \\ \gcd(c,d')=1}} \sum_{\substack{l \in \mathbf{Z} \\ d = lc+d'}} \int_{0+iy}^{1+iy} \frac{e(m\gamma z)}{j_\gamma(z)} e(-nz) dz.$$

The first terms gives $2\delta_{mn}$, for the second term note that

$$j_{\gamma_d(z)} = (cz+d)^k = (c(z+l)+d')^k = j_{\gamma_{d'}}(z+l)$$

and

$$\gamma z = \frac{az+b}{cz+d} = \frac{a}{c} - \frac{1}{c(cz+d)},$$

hence

$$\gamma_d z = \gamma_{d'}(z+l).$$

As we have terms with $z+l$ now, we change variables with $z+l \mapsto z$ to obtain

$$\begin{aligned} a_n &= 2\delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ \gcd(c,d)=1}} \sum_{l\in\mathbf{Z}} \int_{l+iy}^{l+1+iy} \frac{e(m\gamma z)}{j_\gamma(z)} e(-nz) dz \\ &= 2\delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ \gcd(c,d)=1}} \int_{-\infty+iy}^{\infty+iy} \frac{e(m\gamma z)}{j_\gamma(z)} e(-nz) dz \end{aligned}$$

Note that for $c>0$ we have $j_\gamma(z) = (cz+d)^k = c^k(z+\frac{d}{c})^k$ and $\gamma z = \frac{a}{c} - \frac{1}{c^2(z+\frac{d}{c})}$. We change variables with $z + \frac{d}{c} \mapsto z$ and obtain

$$\begin{aligned} a_n &= 2\delta_{mn} + \sum_{\substack{c>0 \\ d \bmod c \\ \gcd(c,d)=1}} \int_{-\infty+iy}^{\infty+iy} c^{-k} z^{-k} e\left(\frac{ma}{c}\right) e\left(-\frac{m}{c^2 z}\right) e\left(-n\left(z-\frac{d}{c}\right)\right) dz \\ &= 2\delta_{mn} + \sum_{c>0} c^{-k} k(m,n,c) \int_{-\infty+iy}^{\infty+iy} z^{-k} e\left(-\frac{m}{c^2 z} - nz\right) dz \end{aligned}$$

We are nearly finished: the following two lemmas give the expression we seek. □

Lemma. *If J_n denotes the n^{th} Bessel function, then*

$$\int_{-\infty+iy}^{\infty+iy} w^{-k} e^{-\mu_1 w - \mu_2 w^{-1}} dw = 2\pi \left(\frac{\mu_1}{\mu_2}\right)^{\frac{k-1}{2}} e^{-ik\frac{\pi}{2}} J_{k-1}(4\pi\sqrt{\mu_1\mu_2}).$$

Proof. Exercise. □

Lemma.

$$\sum_{n=1}^{\infty} n^{-s} R_n(m) = \frac{\sigma_{s-1}(m)}{m^{s-1}\zeta(s)}$$

where $R_n(m) := \sum_{\substack{k \bmod n \\ \gcd(k,n)=1}} e\left(\frac{km}{n}\right)$ *is the Ramanujan sum.*

Proof. See Hardy and Wright's "An introduction into the theory of numbers", theorem 292. □