The j invariant

Sacha Schweizer

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Introduction

We pursue the discussion of modular functions, studying their relationship with homogeneous lattice functions and the Weierstrass $\wp$ function, which gives us the occasion to define elliptic functions and introduce some basic results. We explain the importance of the Eisenstein series - two of them, in particular - in characterizing $\wp$. We then introduce a special modular function, the j invariant, in terms of which any other modular function of weight 0 can be expressed explicitly. j is shown to give a bijection between the fundamental domain and $\mathbb{C}$ mapping infinity to infinity, which allows us to prove an existence theorem for lattices related to elliptic curves, and Picard's Little Theorem about the behavior of entire functions.

Relationship between homogeneous lattice functions and modular functions

Recall The Eisenstein series of order k is a modular form of weight k, defined as

$$G_k(\tau) = \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}\smallsetminus\{(0,0)\}} \frac{1}{(m\tau+n)^k}$$

It can be rewritten as

$$G_k(L) = \sum_{\omega\in L\smallsetminus\{0\}} \frac{1}{\omega^k}$$

for the lattice $L = L_\tau = \tau\mathbb{Z}+\mathbb{Z}$. This more general expression takes any lattice

$$L = \omega_1\mathbb{Z}+\omega_2\mathbb{Z}, \quad \omega_1,\omega_2\in\mathbb{C}, \quad \frac{\omega_1}{\omega_2}\notin\mathbb{R}$$

as argument and defines a *lattice function*, which turns out to be *homogeneous of degree $-k$*: for any $\lambda\in\mathbb{C}$, let λL denote the expansion by λ of L, $\lambda\omega_1\mathbb{Z}+\lambda\omega_2\mathbb{C}$, then

$$\begin{aligned} G_k(\lambda L) &= \sum_{\omega\in\lambda L\smallsetminus\{0\}} \frac{1}{\omega^k} \\ &= \sum_{\omega\in L\smallsetminus\{0\}} \frac{1}{(\lambda\omega)^k} \\ &= \lambda^{-k} G_k(L) \end{aligned}$$

Note that for $\tau\in\mathbb{H}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$[1],

$$L_\tau = \gamma L_\tau = (a\tau+b)\mathbb{Z}+(c\tau+d)\mathbb{Z}$$

For any homogeneous lattice function f of degree $-k$,

$$\begin{aligned} f(L_\tau) &= f(\gamma L_\tau) \\ &= (c\tau+d)^{-k} f(L_{\gamma\tau}) \end{aligned}$$

In other words,

$$f(L_{\gamma\tau}) = (c\tau+d)^k f(L_\tau)$$

[1] As usual, Γ denotes the full modular group $SL_2(\mathbb{Z})$.

so $f(\tau) = f(L_\tau)$ is modular of weight k.

We have other modular forms

$$g_2(\tau) = 60G_4(\tau)$$

$$g_3(\tau) = 140G_6(\tau)$$

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau)$$

of weight 4, 6, and 12, respectively.

Elliptic Functions[2]

Definition A function $f : \mathbb{C} \to \mathbb{C}$ such that

$$f(z+\omega) = f(z), \quad \forall z \in \mathbb{C}, \quad \forall \omega \in L$$

for some lattice L is called *doubly periodic*.
A doubly periodic meromorphic function is called *elliptic*.
Elliptic functions form a subfield of the field of meromorphic functions.

Note that by periodicity, such a function is fully determined on a period parallelogram such as $[0,1)\omega_1 + [0,1)\omega_2$
Furthermore, $L \cong \mathbb{Z} \oplus \mathbb{Z}$ as a group under addition acting on $\mathbb{C}$ by translation, hence $\mathbb{C}/L \cong T^2$, and f is actually a function on a torus.

Theorem An elliptic function without poles (or zeroes) is constant.
Proof If f has no poles, it is entire, and continuous on the closure of any period parallelogram. This is compact, hence f is bounded there. By periodicity, f is (globally) bounded, and by Liouville's theorem, it must be constant.
If f has no zeroes, consider $1/f$. $\square$

[2]This portion has been added to this talk because, for reasons of time, it had been left out from the talk $< 1 >$.

Theorem and Definition The number of zeroes and poles (counted with multiplicity) of a nonzero elliptic function f in a period parallelogram coincide; it is called the *order* of f, and it is always at least 2.
Proof Zeroes and poles of a nonzero meromorphic function are discrete, hence locally finite. Integrate the logarithmic derivative of f along the boundary of a period parallelogram so chosen that no zeroes or poles lie on its boundary (translate if needed). By periodicity, opposite sides cancel and the integral is 0. The first part of the claim then follows by the Residue Theorem. The second part follows by the Residue Theorem applied to f instead, and considering our previous theorem. $\square$

An example of elliptic function

Definition The *Weierstrass $\wp$ function* for a lattice L is defined on $\mathbb{C}$ as

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$$

Theorem $\wp$ is an elliptic function for L of order 2, with double poles at lattice points (without proof[3]).

Theorem $\wp$ possesses the following Laurent expansion around the origin:

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} (2n+1) G_{2n+2} z^{2n}$$

for $\|z\| < min\{\|\omega\| : \omega \in L\}$.

Proof The proof is based on the Taylor expansion of $\frac{1}{(z-\omega)^2}$ and can be found in $< 1 >$, 3. $\square$

[3]For a proof, see [1], 1.6

Theorem $\wp$ satisfies the following differential equation:

$$[\wp'(z)]^2 = 4\wp^3(z) - g_2(L)\wp(z) - g_3(L).$$

Proof Knowledge of the above Laurent expansion of $\wp$ around the origin allows one to conclude that the difference between left-hand and right-hand side of the differential equation is a Taylor series without constant term. By ellipticity, it must be constant, hence zero. For the computational details, see[4] [1], 1.9. □

Remarks 1) This differential equation leads us to ask ourselves whether $\wp$ is fully determined by g_2 and g_3 alone. This is indeed so, since the Eisenstein series are expressible[5] as polynomials in g_2 and g_3. In turn, $\wp$ suffices for representing all elliptic functions[6].
2) $\Delta(L)$ is the *discriminant* of the cubic polynomial *in Weierstrass form* $P(z) = 4z^3 - g_2(L)z - g_3(L)$; the discriminant is a constant multiple of

$$[(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)]^2$$

where z_1, z_2, z_3 are the roots of P, so

$$\Delta(L) \neq 0 \Longleftrightarrow \text{P has distinct roots}$$

Any nondegenerate cubic polynomial can be transformed by affine change of variable to the form $4z^3 - Az - B$. We will see later on that in case the polynomial has distinct roots, there exists a lattice L such that

$$A = g_2(L), \quad B = g_3(L)$$

[4]The numbers in square brackets correspond to the references given in the outline of the seminar, see *General Information*

[5]g_2 and g_3 are constant multiples of E_4 and E_6, respectively, which are shown to generate all modular forms in < 2 >, 4.

[6]See [5], §I.5.

In particular, any elliptic curve can be brought into Weierstrass form.

The j invariant

Definition The j *invariant*[7] is defined as

$$j(\tau) = 1728\frac{g_2^3(\tau)}{\Delta(\tau)}, \quad \tau \in \mathbb{H}.$$

Note j is the ratio of two modular forms of weight 12, hence it is a modular function of weight 0. Since Δ has a simple zero at infinity but vanishes nowhere else, j has a simple pole at infinity and is holomorphic on $\mathbb{H}$.

Let $\widehat{\Gamma\backslash\mathbb{H}}$ denote the closure of the modular fundamental domain with Γ-equivalent sides identified and the point at infinity added. Topologically, $\widehat{\Gamma\backslash\mathbb{H}}$ can be given the structure of the *one-point compactification* of the fundamental domain, which is homeomorphic to the 2-sphere.

Theorem j defines a bijection

$$\widehat{\Gamma\backslash\mathbb{H}} \longrightarrow \mathbb{C} \cup \{\infty\}.$$

Here the codomain is nothing but the Riemann sphere, $\mathbb{C} \cup \{\infty\} \cong \mathbb{CP}^1$.

Proof As seen above,

$$j(i\infty) = \infty$$

It remains to show that on the fundamental domain, j attains every complex value exactly once. In other words, for each $\lambda \in \mathbb{C}$ there exists exactly one $z \in \Gamma\backslash\mathbb{H}$ such that

$$g_2^3(z) - \lambda\Delta(z) = 0.$$

[7]Serre [7] calls it the *modular invariant*. Apostol [1] writes J instead, defines it first in terms of a lattice, leaves out the factor 1728 and calls it *Klein's modular function*. We adopt the notation and definition of Koblitz's [5].

For each λ, the left-hand side of the latter equation defines a modular form f_λ of weight 12. By the valence formula[8] with "$k = 12$", we obtain[9]

$$v_\infty(f_\lambda) + \frac{1}{2}v_i(f_\lambda) + \frac{1}{3}v_\omega(f_\lambda) + \sum_{p\in(\Gamma\backslash\mathbb{H})\smallsetminus\{i,\omega\}} v_p(f_\lambda) = 1.$$

Because $g_2(i\infty)$ is a nonzero complex number while $\Delta(i\infty) = 0$, the term $v_\infty(f_\lambda)$ vanishes. As a modular form, f_λ has no poles, the remaining terms in the valence formula are thus all nonnegative, so we get a decomposition

$$\frac{1}{2}a + \frac{1}{3}b + c = 1, \quad a, b, c \in \mathbb{N}$$

It is easy to check by substitution of the few possible candidates that the solutions are

$$(a, b, c) \in \{(2, 0, 0), (0, 3, 0), (0, 0, 1)\}$$

In all three cases, f_λ vanishes at exactly one point in the fundamental domain. □

Remark $\widehat{\Gamma\backslash\mathbb{H}}$ can be given a complex manifold structure arising in a natural way from the compactification of the quotient by the action of the modular group on the complex manifold $\mathbb{H}$. Then the above bijection is actually an isomorphism of complex manifolds, which would not be the case if we started with an arbitrary homeomorphism (such would be fairly easy to construct).

The next proposition shows how any modular function of weight 0 can be expressed in terms of j.

Theorem Let f be a meromorphic function on $\mathbb{H}$. Then the following are equivalent:

(i) f is a modular function of weight 0

(ii) f is a quotient of modular *forms* of the same weight

(iii) f is a rational function of j.

Proof The implication (ii)⇒(i) is obvious.

To prove (iii)⇒(ii), suppose

$$f(z) = \frac{P(j(z))}{Q(j(z))}$$

for complex polynomials P and Q (not necessarily of the same degree), $Q \neq 0$. P(j(z)) and Q(j(z)) are both modular functions of weight zero holomorphic on $\mathbb{H}$. They have at most a pole at $i\infty$, which we can kill by multiplication with the cusp form Δ, so

$$P(j(z))\Delta^k(z), Q(j(z))\Delta^k(z)$$

are both holomorphic on $\mathbb{H} \cup \{\imath\infty\}$ for some $k \in \mathbb{N}$, and modular of weight $12k$. So

$$f(z) = \frac{P(j(z))\Delta^k(z)}{Q(j(z))\Delta^k(z)}$$

is a ratio of modular forms of the same weight.

To prove (i)⇒(iii), suppose f is modular of weight 0. Set

$$g(z) = \prod_{p\in\Gamma\backslash\mathbb{H}: v_p(f)<0} (j(z) - j(p))^{-v_p(f)} f(z)$$

in order to kill the poles of f on $\mathbb{H}$. g is meromorphic at infinity and modular of weight 0. In particular, there is some $k \in \mathbb{N}$ such that $g(z)\Delta^k(z)$ is a modular form of weight $12k$, which can therefore be written as a linear combination of the form[10]

$$g(z)\Delta^k(z) = \sum_{4n+6m=12k} c_{nm} G_4^n(z) G_6^m(z)$$

[8] See < 2 >, 3.

[9] As in the previous talks, $\omega = -\frac{1}{2} + \frac{\sqrt{3}i}{2}$.

[10] The normalized Eisenstein series E_* considered in < 2 > differ only by a numerical factor from the G_*'s here.

for complex coefficients c_{nm}. Note that

$$j(z) = 1728\frac{(60G_4(z))^3}{\Delta(z)}$$

$$= 1728\frac{G_4^3(z)}{G_4^3(z) - \frac{49}{20}G_6^2(z)},$$

so both $\frac{G_4^3}{\Delta}$ and $\frac{G_6^2}{\Delta}$ have the form $aj + b$. Furthermore, the above linear combination runs over indices m, n that satisfy

$$12|4n + 6m,$$

in particular

$$3|n, \quad 2|m,$$

hence

$$g(z) = \sum_{4n+6m=12k} c_{nm}(\frac{G_4^3(z)}{\Delta(z)})^{\frac{n}{3}}(\frac{G_6^2(z)}{\Delta(z)})^{\frac{m}{2}},$$

and g is actually a polynomial in j. Reversing our definition of g we get an expression for f as a ratio of polynomials in j. □

Remark (contd.) With the appropriate complex manifold structure on $\widehat{\Gamma\backslash\mathbb{H}}$, this theorem shows that the meromorphic functions on the Riemann sphere are exactly the rational functions.

We now come to the promised result with applications in the theory of elliptic functions.

Theorem For each pair of complex numbers A and B such that $A^3 \neq 27B^2$ there exists a lattice $L = \lambda L_z$ such that

$$A = g_2(L), \quad B = g_3(L).$$

Proof Note that

$$\frac{g_3^2}{g_2^3} = \frac{1}{27}(1 - \frac{1728}{j})$$

takes on every value including infinity except $\frac{1}{27}$ on $\mathbb{H}$, since j takes on every complex value there, as seen above. On the other hand, the condition $A^3 \neq 27B^2$ translates into

$$\frac{B^2}{A^3} \neq \frac{1}{27}.$$

There exists thus some $z \in \mathbb{H}$ such that

$$\frac{g_3^2(z)}{g_2^3(z)} = \frac{B^2}{A^3}.$$

Choose $\lambda \in \mathbb{C} \setminus \{0\}$ such that

$$g_2(z) = \lambda^4 A,$$

then

$$g_3^2(z) = \frac{B^2}{A^3}g_2^3(z) = \lambda^{12}B^2,$$

or

$$g_3(z) = \pm\lambda^6 A,$$

In the case of a "−", replace λ by $i\lambda$, so without loss of generality we may assume the case "+"to hold. The result follows by the homogeneity of g_2 and g_3 demonstrated at the beginning of the talk. □

Picard's Little Theorem

Another application of the j invariant consists in the proof of the following well-known theorem from complex analysis, which however stands quite apart from the topics of this seminar.

Theorem Every nonconstant entire function attains every complex value with at most one exception.

Note Nonconstant polynomials take on every complex value (by the Fundamental Theorem of Algebra). The exponential function omits only the value 0.

Proof (Sketch[11]) Without loss of generality, let f be an entire function that omits at least the values 0 and 1728 (which turn out to be $j(\omega)$ and $j(i)$, respectively). Since $\mathbb{H}$ covers the fundamental domain by projection to the quotient by Γ, j maps $\mathbb{H}$ to a Riemann surface with branch points at 0, 1728 and ∞ (imagine"folding" $\mathbb{H}$ elastically onto itself over and over so as to get Γ-translates of the fundamental domain to lie on top of one another: the "problem points" will be at the "vertices"of each translate). This has a multivalued "inverse"(similar to the complex logarithm) that can be composed with f, since by assumption f omits the values 0, 1728 and ∞. By the monodromy theorem, the composition has a single-valued analytical continuation on $\mathbb{C}$, but it only takes values in $\mathbb{H}$, so it is constant (the latter step is more elementary, requiring only Liouville's theorem, and is often given as an exercise in complex analysis books), hence so is f by application of j. □

References

< 1 > Salvatore Bonaccorso: *Modular forms, Eisenstein series and a short introduction to elliptic functions.*

< 2 > Jonas Jermann: *The valence formula and the space of modular forms.*

ssacha@student.ethz.ch

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[11] A complete proof would require knowledge beyond an introductory complex analysis course. A more detailed proof, though still somewhat sketchy, than the one given here is laid out in [1], 2.7.