Theta Functions

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Abstract

Theta functions are introduced, associated to lattices or quadratic forms.

Their transformation property is proven and the conditions discussed under which theta functions are modular forms.

As an application it is shown how theta functions can be used to analyse lattice sphere packings.

The relation to the represantation number of natural numbers as sums of squares is mentioned.

1 Definition of the theta functions

1.1 The theta function of a lattice

Let V be a n-dimensional real vector space equipped with an inner product (a bilinear, symmetric, positive definite form) $\langle \cdot, \cdot \rangle$.

Let μ be an invariant measure on V, i.e. for any measurable subset $U \subset V, \forall v \in V : \mu(v + U) = \mu(U)$ and $\forall \lambda \in \mathbb{R} : \mu(\lambda U) = |\lambda|^n \mu(U)$. Furthermore μ is to be scaled so that for a basis $(\varepsilon_1, \ldots \varepsilon_n)$ of V which is orthonormal with respect to $\langle \cdot, \cdot \rangle$: $\mu(\{\sum_j \lambda_j \varepsilon_j \mid 0 \leq \lambda_j \leq 1\}) = 1$.

Definitions A *lattice* Γ is a subset of V which can be written in the form

$$\Gamma = e_1\mathbb{Z} + \ldots + e_n\mathbb{Z}$$

where $(e_1, \ldots e_n)$ is a basis of V, which is then called a *basis of* Γ.

Equivalently it can be defined as a discrete subgroup which does not lie in a hyperplane.

The *volume* of Γ is $\mathrm{vol}(\Gamma) := \mu(V/\Gamma) = \mu(\{\sum_j \lambda_j e_j \mid 0 \leq \lambda_j \leq 1\})$.

The *dual lattice* Γ' of Γ is a subset of the dual space V' defined by

$$h \in \Gamma' \quad :\Leftrightarrow \quad \forall g \in \Gamma : h(g) \in \mathbb{Z}$$

To the lattice Γ one assigns the *theta function* $\theta_\Gamma : \mathbb{H} \to \mathbb{C}$,

$$\tau \mapsto \sum_{g \in \Gamma} e^{\pi i \tau \langle g, g \rangle} \tag{1}$$

Remarks To verify that Γ' is itself a lattice, let $(e_1', \ldots, e_n') \subset V'$ denote the dual basis of $(e_1, \ldots e_n)$, i.e. $e_j(e_k) = \delta_{jk}$ (Kronecker delta). $(e_1', \ldots, e_n')$ is a basis of V' and a subset of Γ' and any $h \in \Gamma'$ can be decomposed to $h = \sum_j h(e_j) \cdot e_j' \in e_1'\mathbb{Z} + \ldots + e_n'\mathbb{Z}$. So $(e_1', \ldots e_n')$ is a basis of Γ'.

V can be identified with V' via the inner product: $v \mapsto (w \mapsto \langle v, w \rangle)$. Γ' will also denote the image of Γ under this isomorphism, which is another lattice in V.

Lemma (1) converges uniformly on every compact subset and it converges absolutely. Thus θ_Γ is holomorphic.

Proof (Sketch) Since $\tau \in \mathbb{H} \Rightarrow \mathrm{Re}\, i\tau < 0$ and $\langle g, g \rangle > 0 \ \forall g \neq 0$, the exponent's real part is negative. The Lemma can thus be proven by finding a bound for the number of lattice points with $\langle g, g \rangle \leq x$ which is polynomial in x.

1.2 Theta function of a quadratic form

Definitions A *quadratic form* is a map $S : V^2 \to \mathbb{R}$ so that $(v, w) \mapsto S[v + w] - S[v] - S[w]$ is an inner product.

Let $\mathrm{Pos}(n, \mathbb{R})$ be the set of symmetric, positive definite $n \times n$-matrices and let $S \in \mathrm{Pos}(n, \mathbb{R})$. Such an S defines a quadratic form by $v \mapsto S[v]$ where $S[v] := v^\top S v$. All quadratic forms are given in such a way.

One associates the theta function $\theta_S : \mathbb{H} \to \mathbb{C}$ to S, which is given by

$$\tau \mapsto \sum_{g \in \mathbb{Z}^n} e^{\pi i \tau S[g]}$$

Remark Let A be the Gram matrix of $(e_1, \dots e_n)$ with respect to $\langle \cdot, \cdot \rangle$, i.e. $A_{ij} = \langle e_i, e_j \rangle$.

It follows that $\theta_\Gamma = \theta_A$ because:

$$g = \sum_{j=1}^{n} \lambda_j e_j \quad \Rightarrow \quad \langle g, g \rangle = (\lambda_1 \ \cdots \ \lambda_n) A \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

So defining a theta function via a lattice or a quadratic form is just a difference of perspective.

Lemma $\mu(V/\Gamma) = \sqrt{\det(A)}$

Proof Let $B := (e_1 \ \cdots \ e_n)^\top$ be a $n \times n$ matrix the rows of which are the basis vectors written in terms of a basis orthonormal with respect to $\langle \cdot, \cdot \rangle$. B is called a *generator matrix* of the lattice.

$\det(B) = \mu(\{\sum_j \lambda_j e_j \mid 0 \le \lambda_j \le 1\})$ and

$$B^\top B = A \Rightarrow \det(A) = \det(B^\top)\det(B) = \det(B)^2 = \mu(V/\Gamma)^2$$

because the absolute value of the determinant of a matrix is the volume of the parallelepiped spanned by its row vectors. □

Lemma The inverse matrix A^{-1} is a Gram matrix of the dual lattice Γ'.

Proof Let $e'_j = A^{-1} e_j$. Then $(e'_1, \dots, e'_n)$ is the dual basis (which is a basis of Γ') because $\langle e'_j, e_k \rangle = (A^{-1} e_j)^\top A e_k = e_j^\top e_k = \delta_{jk}$.

So indeed $\langle e'_j, e'_k \rangle = (A^{-1} e_j)^\top A (A^{-1} e_k) = e_j A^{-1} e_k$. □

2 The Transformation Formula

Theorem Let Γ be a lattice. Then

$$\forall \tau \in \mathbb{H} : \quad \theta_{\Gamma'}\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \mu(V/\Gamma)\theta_\Gamma(\tau) \tag{2}$$

where that branch of $\sqrt{}$ is to be chosen that maps positive reals to positive reals.

To prove the theorem one requires the following

Lemma (Poisson formula) A bounded function $f : V \to \mathbb{R}$ is called *rapidly decreasing* if it can be partially differentiated arbitrarily often and all these derivatives are bounded.

The *Fourier transform* $\widehat{f}$ of such a function is defined as

$$\widehat{f} : V' \to \mathbb{R}, \quad h \mapsto \int_V e^{-2\pi i h(g)} f(g)\,\mathrm{d}\mu(g)$$

For any such f and lattice Ω the Poisson formula states that

$$\mu(V/\Omega) \sum_{g\in\Omega} f(g) = \sum_{h\in\Omega'} \widehat{f}(h)$$

Proof Use a basis of Ω to identify V with the $\mathbb{R}^n$; Ω is mapped to $\mathbb{Z}^n$, a lattice with volume 1. Identify μ with the product measure $\mathrm{d}x_1 \ldots \mathrm{d}x_n$. Then use $\mathbb{R}^{n\prime} = \mathbb{R}^n$ and $\mathbb{Z}^{n\prime} = \mathbb{Z}^n$ and notice that the fourier transform as defined above is identified with the known fourier transform (using the right scaling of μ with respect to $\langle\cdot,\cdot\rangle$).

So this more general form of the Poisson formula follows from the classical

$$\sum_{g\in\mathbb{Z}^n} f(g) = \sum_{h\in\mathbb{Z}^n} \widehat{f}(h)$$

$\square$

Proof of the Transformation Formula By the identity theorem, for two holomorphic functions defined on a open connected set – such as $\mathbb{H}$ – to be equal it is sufficient that they agree on a subset which has an accumulation point contained in that subset. So it is sufficient to prove the transformation formula for $\tau = iy$ with $y \in \mathbb{R}^+$, that is to prove

$$y^{n/2}\mu(V/\Gamma) \sum_{g\in\Gamma} e^{-\pi y\langle g,g\rangle} = \sum_{h\in\Gamma'} e^{-\pi y^{-1}\langle h,h\rangle}$$

Let $f : V \to \mathbb{R},\ g \mapsto e^{-\pi\langle g,g\rangle}$. f is rapidly decreasing and $\widehat{f} = f$ if you identify V with V' using $\langle\cdot,\cdot\rangle$. This can be proven by doing the same identifications as in the proof of the Poisson formula, thus reducing the claim to $\widehat{e^{-\pi\|x\|^2}} = e^{-\pi\|x\|^2}$ for $x \in \mathbb{R}^n$, a known analytic result.

Let $\Omega = y^{1/2}\Gamma$. Then $\Omega' = y^{-1/2}\Gamma'$ and $\mu(V/\Omega) = y^{n/2}\mu(V/\Gamma)$.

The Poisson formula, applied to f and Ω yields

$$\begin{aligned} y^{n/2}\mu(V/\Gamma) \sum_{g\in y^{1/2}\Gamma} e^{-\pi\langle g,g\rangle} &= \sum_{h\in y^{-1/2}\Gamma'} e^{-\pi\langle h,h\rangle} \\ \Leftrightarrow y^{n/2}\mu(V/\Gamma) \sum_{g\in\Gamma} e^{-\pi y\langle g,g\rangle} &= \sum_{h\in\Gamma'} e^{-\pi y^{-1}\langle h,h\rangle} \end{aligned}$$

And that was to be proven. □

3 The theta functions as modular forms

The transformation property of θ_Γ is similar, but not equal to that of a modular form. What conditions has Γ to satisfy for θ to be a modular form?

If $\langle g, g\rangle$ is an even integer $\forall g \in \Gamma$, say, for a given g, $\langle g, g\rangle = 2c$, then

$$e^{\pi i(\tau+1)\langle g,g\rangle} = e^{2\pi i(\tau+1)c} = e^{2\pi i\tau c} = e^{\pi i\tau\langle g,g\rangle}$$

i.e. θ would have period 1:

$$\theta_\Gamma(\tau + 1) = \theta_\Gamma(\tau) \quad (3)$$

From the matrix point of view this means that the diagonal entries are even; such matrices are called *even*.

If Γ is *selfdual*, i.e. $\Gamma = \Gamma'$, its theta function satisfies the modified transformation property

$$\theta_\Gamma\left(-\frac{1}{\tau}\right) = \left(\frac{\tau}{i}\right)^{\frac{n}{2}} \mu(V/\Gamma)\theta_\Gamma(\tau) \quad (4)$$

Lemma If Γ is selfdual and even then $8|n$.

Proof The modular group is generated by the matrices $T = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$.

To prove the lemma, one makes use of $(ST)^3 i = i$:

Choose $\tau = i$ and apply alternatingly (3) and (4):

$$\begin{aligned}
\theta_\Gamma(i) &\overset{(3)}{=} \theta_\Gamma(i+1) \\
&\overset{(4)}{=} \left(\frac{i+1}{i}\right)^{n/2} \theta_\Gamma\left(-\frac{1}{i+1}\right) \\
&= (1-i)^{n/2}\theta_\Gamma\left(\frac{i-1}{2}\right) \\
&\overset{(3)}{=} (1-i)^{n/2}\theta_\Gamma\left(\frac{i+1}{2}\right) \\
&\overset{(4)}{=} \left(\frac{1}{i}\frac{i+1}{2}(1-i)\right)^{n/2} \theta_\Gamma\left(-\frac{2}{i+1}\right) \\
&= (-i)^{n/2}\theta_\Gamma(i-1) \\
&\overset{(3)}{=} (-i)^{n/2}\theta_\Gamma(i)
\end{aligned}$$

Because $\theta_\Gamma(i)$ is a sum of positive terms it is non-zero. So $(-i)^{n/2} = 1 \Rightarrow 8|n$ follows. □

Lemma $\Gamma = \Gamma' \quad \Leftrightarrow \quad \det(A) = 1$ and $\forall j,k : A_{jk} \in \mathbb{Z}$.

Proof "$\Rightarrow$": $\Gamma = \Gamma' \Rightarrow \sqrt{\det(A)} = \sqrt{\det(A^{-1})} \Leftrightarrow \det(A) = 1$. And $A_{jk} = \langle e_j, e_k\rangle \in \mathbb{Z}$.

"$\Leftarrow$": Let $g, h \in \Gamma$, then $\langle g, h\rangle = g^\top A h \in \mathbb{Z} \Rightarrow \Gamma \subset \Gamma'$. A^{-1} is itself a matrix with determinant 1 and integral entries, so switching the roles of $\Gamma = \Gamma''$ and Γ' one deduces $\Gamma' \subset \Gamma \Rightarrow \Gamma = \Gamma'$. □

Corollary The theta function of a selfdual, even lattice Γ is a modular form of weight $\frac{n}{2}$.

Proof θ_Γ is 1-periodic and using the two prior lemmas the transformation property (4) of a selfdual lattice becomes

$$\theta_\Gamma\left(-\frac{1}{\tau}\right) = \tau^{\frac{n}{2}}\theta_\Gamma(\tau)$$

□

4 Sphere packings

A *sphere packing* is a set of non-overlapping balls of same radius in the $\mathbb{R}^n$. Its *density* is the average share of volume covered by the balls.

The final task is to find the densest packing and prove it is the densest.

One important family of packings are the *lattice packings*, the midpoints of the balls of which form a lattice in the $\mathbb{R}^n$.

To find the densest sphere packing for a given lattice one would naturally choose the greatest possible radius so that the balls do not overlap. That is half of the smallest distance between any two distinct lattice points, and because the lattice is invariant under translation by one of its vectors, this distance is equal to the smallest distance of any one lattice point to the origin. That equals $\sqrt{\min_{g\in A}\langle g, g\rangle}$.

Theta functions can be used as a tool to find this distance and determine the kissing number: The *kissing number* of a lattice packing is the number of balls which touch any fixed ball, e.g. the ball around the origin.

Remark Let $r_\Gamma(m) = \#\{g \in \Gamma \mid \langle g, g\rangle = m\}$. The sum (1)

$$\tau \mapsto \sum_{g\in\Gamma} e^{\pi i\tau\langle g,g\rangle}$$

defining θ_Γ can then be rewritten as

$$\tau \mapsto \sum_{m=0}^{\infty} r_\Gamma(m)e^{\pi i\tau m} = 1 + \sum_{m=1}^{\infty} r_\Gamma(m)e^{\pi i\tau m}$$

Since Γ is an even lattice $r_\Gamma(2m + 1) = 0$ the above is equivalent to

$$\tau \mapsto 1 + \sum_{m=1}^{\infty} r_\Gamma(2m)e^{2\pi i\tau m} \tag{5}$$

Here the smallest m for which $r_\Gamma(m)$ is non-zero would be the square of the minimal distance and $r_\Gamma(m)$ itself the kissing number.

In small dimensions this can be calculated explicitly:

The case $\mathbf{n = 8}$ The vector space of modular functions of weight 4 is one dimensional. So the theta function of an eight dimensional lattice is a multiple of the Eisenstein series G_4. Comparing the constant coefficient it turns out that $\theta_\Gamma = E_4$, where E_4 is the normed Eisenstein series

$$E_4(\tau) = 1 + 240\sum_{m=1}^{\infty}\sigma_3(m)e^{2\pi i\tau m} = \theta_\Gamma(\tau)$$

Two functions are equal iff their Fourier coefficients are equal, so (see (5))

$$\Rightarrow \quad r_\Gamma(m) = 240\sigma_3(m)$$

where $\sigma_k(m)$ is the weighted sum of divisors of m, i.e. $\sigma_k(m) = \sum_{\ell\mid m}\ell^k$. This can be calculated: $\sigma_3(1) = 1$, so the minimal distance is $\sqrt{2}$ and the kissing number 240.

The case $\mathbf{n = 16}$ The vector space of modular functions of weight 8 is still one dimensional. Likewise $\theta_\Gamma = E_8$ and

$$r_\Gamma(m) = 480\sigma_7(m)$$

Again the minimal distance is $\sqrt{2}$. The kissing number is 480.

The case $\mathbf{n = 24}$ The vector space of modular functions of weight 12 is, alas, two dimensional and one of the basis elements is a cusp form the coefficients of which cannot be so easily calculated.

The results of applying theta functions to 24-dimensional lattices are therefore, while important, not that explicit as in the cases above.

5 Sums of squares

If we chose $S = I$, the identity matrix, or equivalently $\Gamma = \mathbb{Z}^n$, the resulting quadratic form maps g to $g_1^2 + \ldots + g_n^2$. Its theta function θ_I

$$\tau \mapsto \sum_{k=1}^{\infty} r_n(k) e^{\pi i k \tau}$$

(where $r_n(k)$ is the number of ways to write k as a sum of n squares) is not a modular form with respect to the whole of the modular group, because the identity is not even.

However, θ_I has period 2 and can be considered as a modular form with respect to the subgroup of the modular group generated by the matrices $T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Analysing this subgroup can lead to expressions for the number of ways to represent natural numbers as the sums of squares.

Literature

[1] Jean-Pierre Serre, *A Course in Arithmetic.* Graduate Texts in Mathematics, Springer, 1973.

[2] Max Koecher, Alois Krieg, *Elliptische Funktionen und Modulformen.* Springer, 1998.

[3] Noam D. Elkies, *Lattices, Linear Codes and Invariants.* Notices of the AMS, Volume 47, Number 10 (November 2000).