The valence formula and the space of modular forms

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Introduction

Last time we have seen the definition of modular forms and some of their basic properties. Now our goal is to find a characterization of modular forms (and cusp forms).

And indeed at the end of this paper (Prop. 4.1, 4.2 and 4.3) we will find one, involving E_4 and E_6 - the normalized Eisenstein series of weight 4 and 6. Namely that $M_k(\Gamma)$ - the space of modular forms of weight k - is finite dimensional with the basis:

$$\mathcal{B}_k := \{E_4^i E_6^j | i, j \geq 0,\ 4i + 6j = k\}$$

Before we proof this main result, we will introduce modular functions - a generalization of modular forms. The main ingredient of this paper will then be a more general proposition about modular functions, the valence formula or $\frac{k}{12}$-proposition.

Using the valence formula on modular forms we will find the mentioned characterization of $M_k(\Gamma)$.

The proof of the valence formula uses complex analysis, so we will start with a short review of some needed facts from complex analysis (mainly the argument principle).

1 Facts from complex analysis

1.1 Proposition (Argument principle)**.** *Let $0 \neq f$ be a meromorphic function on D, D simply connected, γ a closed rectifiable curve (CRC) in D not passing any zeroes or poles of f $\Rightarrow$*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz = \sum_{p\in D} Res\left(\frac{f'}{f},p\right)\cdot\chi(\gamma,p) = \sum_{p\in D} v_p(f)\cdot\chi(\gamma,p) \tag{1}$$

$v_p(f) = Res\left(\frac{f'}{f},p\right)$ denotes the **order** of p wrt f, i.e. it is the n such that $\frac{f(z)}{(z-p)^n}$ is holomorphic and not zero in p.

$$v_p(f)\begin{cases} > 0 & \text{if p is a zero}\\ < 0 & \text{if p is a pole}\\ = 0 & \text{otherwise}\end{cases}$$

$\chi(\gamma,p) := \frac{1}{2\pi i}\oint_\gamma \frac{1}{\xi-p}d\xi$ is the **winding number** of p wrt γ.

Remark. Since $\chi(\gamma,\cdot)$ is zero except for finitely many zeroes and poles the sum above is a finite sum.

1.2 Proposition (Integration over small arcs)**.** *Let γ be an arc of radius $\epsilon > 0$ and angle $\Delta\varphi$, then:*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz \xrightarrow{\epsilon\to 0} \frac{1}{2\pi}v_p(f)\cdot\Delta\varphi \tag{2}$$

2 Modular functions

2.1 Definition. Let $k \in \mathbb{Z}$ and $f : \mathbb{H} \to \mathbb{C}$. f is called **modular function of weight** k for $\Gamma = SL_2(\mathbb{Z})$ if:

(a) f satisfies the *transformation property for k*:

$$f(\gamma z) = (cz+d)^k f(z) \quad \forall\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma$$

(b) f is meromorphic on $\mathbb{H}$

(c) f is meromorphic at ∞ (see below)

2.2 Definition ($\tilde{f}$)**.** Let $0 \neq f : \mathbb{H} \to \mathbb{C}$ be a function satisfying (a) and (b). Then f possesses a Fourier expansion (for further details see: Freitag / Busam, Funktionentheorie 1, Berlin 2000, p. 149):

$$f(z) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n z} = \sum_{n\in\mathbb{Z}} a_n q^n,\ \ q = e^{2\pi i z}$$

We define $\tilde{f} : D = \{q \in \mathbb{C} \mid |q| < 1\} \to \mathbb{C}$ (where $q \in D$ is threated as a function argument now) by:

$$\tilde{f}(q) := \sum_{n\in\mathbb{Z}} a_n q^n \quad \left(\Rightarrow\ \ f(z) = \tilde{f}(e^{2\pi i z})\right)$$

$\tilde{f}$ is a meromorphic on $ann(\epsilon,1) = \{z \in \mathbb{C} \mid \epsilon < |z| < 1\}$ for each $\epsilon > 0$ and can also be written (defined) as: $\tilde{f}(q) = f\left(\frac{Log(q)}{2\pi i}\right)$. Since $f(z+1) = f(z)$, this definition of $\tilde{f}$ does not depend on the branch of logarithm.

2.3 Definition. Such an f **is meromorphic at** ∞ if $\tilde{f}$ is meromorphic at 0. This means: $\exists m > -\infty,\ R > 0:\ a_n = 0$ for $n < m$ (ie. 0 is a at most a pole) and $\tilde{f}$ is holomorphic in $ann(0,R) = \{z \in \mathbb{C} | z \neq 0,\ |z| < R\}$.

2.4 Definition (Repetition)**.** A modular function is called **modular form** (of weight k for Γ) if it is holomorphic and holomorphic at ∞ (this means that $\tilde{f}$ is holomorphic at 0, ie. $a_n = 0 \quad \forall n < 0,\ \mathbf{f}(\infty) := \mathbf{a_0}$). The set of such functions is denoted $M_k(\Gamma)$. It forms a $\mathbb{C}$-vector space.

2.5 Definition. A modular form f with $f(\infty) = a_0 = \tilde{f}(0) > 0$ is called a **cusp form** (of weight k for Γ). The set of such functions is denoted $S_k(\Gamma)$. It forms a $\mathbb{C}$-vector space.

2.6 Definition. Let $0 \neq f$ be a modular function. The **order of** ∞ wrt f is defined as: $v_\infty(f) := v_0(\tilde{f})$

2.1 Lemma (Basic properties)**.**

(a) f *a modular function of odd weight* $\Rightarrow\ f \equiv 0$

(b) $0 \neq f$ *a modular function* $=> v_p(f) = v_{gp}(f) \quad \forall g \in \Gamma$

(c) $f \in M_k(\Gamma) \Rightarrow v_p(f) \geq 0 \quad \forall p \in \mathbb{H}$ *and for* $p = \infty$

(d) $f \in S_k(\Gamma) \Leftrightarrow (f \in M_k(\Gamma) \wedge v_\infty(f) > 0)$

Proof. (a): $g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma,\ f(z) = f(gz) = (-1)^k f(z) \overset{\text{(k odd)}}{=} -f(z) \ \Rightarrow\ f \equiv 0.$
(b): Let $g \in \Gamma$, $m = v_{gp}(f) = v_p(fog)$ is such that $\frac{f(gz)}{(z-p)^m} = (cz+d)^k \cdot \frac{f(z)}{(z-p)^m}$ is holomorphic and non zero in p. Since $cp + d \neq 0$ on $\mathbb{H}$ the $(cz+d)^k$ term doesn't matter for the order and m is exactly $v_p(f)$.
(c) and (d) follow immediately from the definition of $M_k(\Gamma)$ and $S_k(\Gamma)$. ☐

2.2 Lemma. $0 \neq f$ *a modular function* $\Rightarrow \exists R < \infty$*: f is holomorphic and non zero for* $\mathrm{Im}(z) > R$*.*

Proof. f is meromorphic at $\infty \Rightarrow \exists R_1 > 0$: $\tilde{f}$ is holomorphic in $ann(0,R_1) = \{z \in \mathbb{C} | z \neq 0,\ |z| < R_1\}$.
$f \neq 0 \Rightarrow \tilde{f} \neq 0 \Rightarrow 0$ can't be an accumulation point of the zeroes of $\tilde{f}$ (complex analysis) $\Rightarrow\ \exists R_2 > 0$: $\tilde{f}$ is holomorphic and non zero in $ann(0,R_2) \Rightarrow f = \tilde{f}oq$ is holomorphic and non zero in $z \in \mathbb{H}$ if: $|q(z)| < R_2 \Leftrightarrow |e^{2\pi i z}| = e^{-2\pi \mathrm{Im}(z)} < R_2 \Leftrightarrow -2\pi \mathrm{Im}(z) < log(R_2) \Leftrightarrow \mathrm{Im}(z) > \frac{log(R_2)}{2\pi} =: R$ ☐

2.7 Definition. For $k \geq 4$, k even the **normalized Eistenstein series** are defined as:

$$E_k := \frac{1}{2\zeta(k)} G_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

B_k are the Bernoulli numbers, defined by: $\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},\ \ \sigma_{k-1}(n) := \sum_{d|n} d^{k-1}$

Remark. $E_k \in M_k(\Gamma)$ because $G_k \in M_k(\Gamma)$ (see last lecture). And for k=4,6 the series are given as:

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \qquad\qquad E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

2.8 Definition. The **discriminant** Δ is defined as: $\Delta(z) := \frac{(2\pi)^{12}}{1728} \left(E_4(z)^3 - E_6(z)^2\right)$

Note. $\Delta \in S_{12}(\Gamma)$, because: $E_4(z)^3 \neq E_6(z)^2$ lie both in $M_{12}(\Gamma)$ and the E_k's and their products are normalized $(f(\infty) = 1)$, so the difference above vanishes at ∞.

3 The valence formula

3.1 Proposition (valence formula or $\frac{k}{12}$-Proposition)**.** *Let $0 \neq f$ be a modular function of weight k for* Γ, $\omega := -\frac{1}{2} + \frac{i\sqrt{3}}{2} \Rightarrow$

$$\boxed{v_\infty(f) + \frac{1}{2}v_i(f) + \frac{1}{3}v_w(f) + \sum_{\substack{p\in\Gamma\backslash\mathbb{H}\\ p\neq i,\omega}} v_p(f) = \frac{k}{12}} \tag{3}$$

Remark. Since $0 \neq f$ is mermomorphic it only has a finite number of representants of zeroes and poles in F and by Lemma 2.1 (b) it follows that the order does not depend on the representants, so the sum is well defined.

Proof. The idea of the proof is to count the (order) of the zeroes and poles in $\Gamma\backslash\mathbb{H}$ by integrating the logarithmic derivative $\frac{f'}{f}$ of f around the boundary of the fundamental domain F_Γ and using the argument principle.

More precisely, we need an approximation first and start with a curve as shown in Figure 1. The curve is choosen in such a way that it contains exactly one representant in $\Gamma\backslash\mathbb{H}$ of each pole and zero, except i, ω (and $-\overline{\omega} \equiv \omega$) which are kept outside.

By Lemma 2.2 f is holomorphic and non zero for $\mathrm{Im}(z) > R$. Therefore we can choose the the top line from $H = \frac{1}{2} + iT$ to $A = \frac{1}{2} + iT$ (with $T > R$) in such a way that f doesn't have any poles or zeroes on or on top of HA. The rest of the contour follows the boundary of F_Γ with a few exceptions:

For each zero or pole $P \neq i, \omega$ on the boundary, we simply circle around it with a small enough radius and the other way round for the congruent point on the other side of the boundary (this way we will only count the point once). This procedure is illustrated for two such points P and Q in Figure 1.

So far we still followed the boundary of F_Γ (modulo Γ) but since we don't want to include i and ω we also have to circle around those points with a small enough radius ϵ (and the *same* way for $-\overline{\omega} \equiv \omega$). In the end we get a CRC $\gamma(t) \equiv \gamma_\epsilon(t)$ with the properties mentioned above:

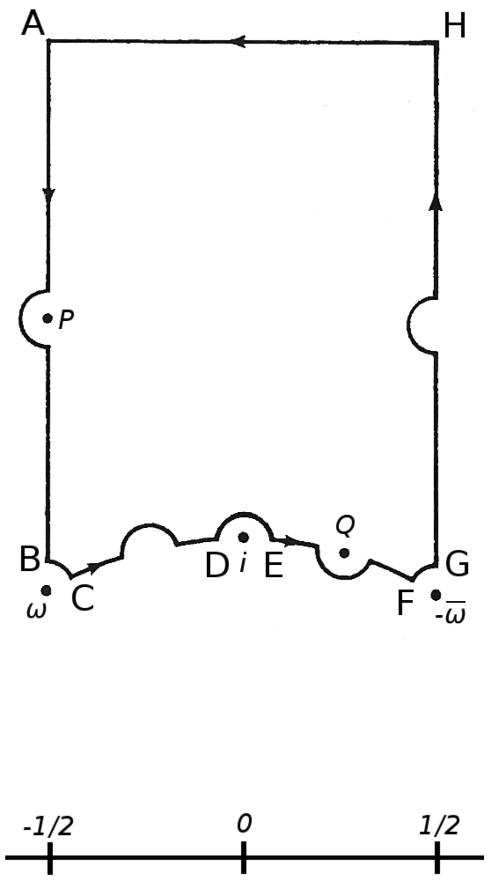

Figure 1: Contour of F_Γ

$\chi(\gamma, p) = 1$ for all $p \in \Gamma\backslash\mathbb{H}, p \neq i, \omega$ and $\chi(\gamma, p) = 0$ otherwise. By the Argument principle (Prop 1.1) applied to f and γ we get:

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}dz \overset{(1.1)}{=} \sum_{p\in\mathbb{H}} v_p(f)\cdot\chi(\gamma,p) = \sum_{\substack{p\in\Gamma\backslash\mathbb{H}\\ p\neq i,\omega}} v_p(f) \tag{4}$$

On the other hand we can evaluate the integral in (4) section by section:

1. $\int_{AB} + \int_{GH} = 0$ $(AB \ni f(\gamma(t)+1)) = f(\gamma(t)) \in GH)$.
2. $\int_{HA=:-\varrho}$: $\varrho(t) = t + iT,\ t \in [-\frac{1}{2}, \frac{1}{2}]$, we take a look at $\tilde{\varrho}, \tilde{f}$:

   $$\tilde{\varrho(t)} = q(\varrho(t)) = e^{2\pi i(t+iT)} = e^{-2\pi T} \cdot e^{2\pi i t},\ t \in [-\frac{1}{2}, \frac{1}{2}]$$
   $$\Rightarrow \tilde{\varrho(t)} \text{ is a circle of radius } e^{-2\pi T}$$
   $$f(z) = \tilde{f} o q(z) \Rightarrow f'(z) = \frac{\mathrm{d}\tilde{f}}{\mathrm{d}q}(q)\frac{\mathrm{d}q}{\mathrm{d}z} \Rightarrow \frac{f'(z)}{f(z)}dz = \frac{\tilde{f}'(q)}{\tilde{f}(q)}dq$$
   $$\Rightarrow \int_{HA=-\varrho} \frac{f'(z)}{f(z)}dz \overset{(q=e^{2\pi i z})}{=} -\oint_{\tilde{\varrho}=q(\varrho)} \frac{\tilde{f}'(q)}{\tilde{f}(q)}dq \overset{(1.1)}{=} -\sum_{q \in B} v_p(\tilde{f}) \cdot \chi(\varrho, p) = -v_0(\tilde{f}) = -v_\infty(f)$$

3. $\int_{BC} + \int_{DE} + \int_{FG}$: Since the 3 sections are all arcs (integrated clockwise!) of a small radius ϵ we can use Prop 1.2 to calculate the limit ($\epsilon \to 0$) of the integrals:

   $$\int_{BC} + \int_{DE} + \int_{FG} \xrightarrow{\epsilon \to 0} -\frac{1}{2\pi i}\left(v_w(f) \cdot \Delta\phi_\omega + v_i(f) \cdot \triangle\phi_i + v_{-\overline{\omega}}(f) \cdot \Delta\phi_{-\overline{\omega}}\right)$$
   $$\left(\Delta\phi_{-\overline{\omega}} = \Delta\phi_\omega = \frac{\pi}{3},\ \Delta\phi_i = \pi\right)$$
   $$= -\left(\frac{v_w(f)}{3} + \frac{v_i(f)}{2}\right)$$

4. $\int_{CD} + \int_{DF}$: We will use a lemma to finnish the last part of the proof. It remains to show that:

   $$\frac{1}{2\pi i}\left(\int_{CD} \frac{f'(z)}{f(z)}dz + \int_{DF} \frac{f'(z)}{f(z)}dz\right) \xrightarrow{\epsilon \to 0} \frac{k}{12} \tag{5}$$

   3.1 Lemma. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,\ c \neq 0,\ 0 \neq f$ *modular function of weight* k, ϱ *a rectifiable curve in* $\mathbb{H}$ $\Rightarrow$

   $$\int_\varrho \frac{f'(z)}{f(z)}dz \ - \int_{g\varrho} \frac{f'(z)}{f(z)}dz = -k\int_\varrho \frac{1}{z+\frac{d}{c}}dz \tag{6}$$

   Proof (Lemma).

   $$f(gz) = (cz+d)^k f(z) \Rightarrow f'(gz)\frac{\mathrm{d}gz}{\mathrm{d}z} = f'(z)(cz+d)^k + ck(cz+d)^{k-1}f(z)$$
   $$\Rightarrow \frac{f'(gz)}{f(gz)}dgz = \frac{f'(z)}{f(z)}dz + \frac{ck}{cz+d}dz$$
   $$\Rightarrow \int_\varrho \frac{f'(z)}{f(z)}dz - \int_{g\varrho} \frac{f'(z)}{f(z)}dz = \int_\varrho \frac{f'(z)}{f(z)}dz - \int_\varrho \frac{f'(gz)}{f(gz)}dgz = -k\int_\varrho \frac{c}{cz+d}dz$$

   □

   In our case we have: $g = S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \varrho = CD,\ g\varrho = FE,\ f = f\ \Rightarrow$

   $$\frac{1}{2\pi i}\left(\int_{CD} \frac{f'(z)}{f(z)}dz + \int_{EF} \frac{f'(z)}{f(z)}dz\right) = \frac{1}{2\pi i}\left(\int_\varrho \frac{f'(z)}{f(z)}dz - \int_{g\varrho} \frac{f'(z)}{f(z)}dz\right)$$
   $$\overset{(Lemma\ 3.1)}{=} -\frac{k}{2\pi i}\int_\varrho \frac{1}{z}dz \xrightarrow{\epsilon \to 0} -\frac{k}{2\pi i}\int_{\left(\substack{\text{arc from } \omega \\ \text{to } i \text{ around } 0}\right)} \frac{1}{z}dz$$
   $$\overset{(z=e^{i\Theta})}{=} -\frac{k}{2\pi}\int_{\Theta(i)\equiv\frac{\pi}{2}}^{\Theta(\omega)\equiv\frac{2\pi}{3}} d\Theta = \frac{k}{2\pi}\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \frac{k}{12}$$

This completes the proof. □

4 The space of modular forms

4.1 Proposition. *Let $k \in \mathbb{Z}$*

(a) $M_0(\Gamma) = \mathbb{C}$, *"Modular forms of weight 0 are constants."*

(b) For $k < 0,\ k = 2 : M_k(\Gamma) = 0$

(c) For $k = 4, 6, 8, 10, 14 : M_k(\Gamma) = \mathbb{C}E_k$,
"$M_k(\Gamma)$ is one dimensional, generated by the normalized Eisenstein series E_k."

(d) For $k < 12,\ k = 14 : S_k(\Gamma) = 0,\ S_{12}(\Gamma) = \mathbb{C}\Delta$ *and for* $k > 14 : S_k(\Gamma) = \Delta M_{k-12}(\Gamma)$,
"Δ is the first non trivial cusp form."

(e) For $k > 2 : M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}E_k$

Proof. Note that that the proposition is about modular *(*forms), by Lemma 2.1 (c) we have: $v_p(f) \geq 0\ \ \forall p \in \mathbb{H}$ and for $p = \infty$. Now we apply Prop. 3.1, LHS and RHS refers to the left and right hand side of equation (3):

(a) $f \in M_0(\Gamma) \ \Rightarrow\ \exists c \in \mathbb{C} : f(z) - c \in M_0(\Gamma)$ has a zero
$\Rightarrow \begin{cases} f - c \equiv 0 \\ RHS \equiv 0\ \wedge\ LHS > 0 \ \ \text{Contradiction!} \end{cases} \ \Rightarrow\ f \equiv c$

(b) $k < 0 \ \Rightarrow\ (LHS > 0 \wedge RHS < 0)$ Contradiction!
$k = 2 \ \Rightarrow\ (RHS = \frac{1}{6}$ but there is no way to get $LHS = \frac{1}{6}$ Contradiction!)

(c) For the mentioned k's we have only one possible choice:
$k = 4 \ \Rightarrow\ v_w(f) = 1$
$k = 6 \ \Rightarrow\ v_i(f) = 1$
$k = 8 \ \Rightarrow\ v_w(f) = 2$
$k = 10 \ \Rightarrow\ v_i(f) = v_w(f) = 1$
$k = 14 \ \Rightarrow\ v_i(f) = 2, v_w(f) = 1$
$0 \neq f_1, f_2 \overset{\text{(same zeroes)}}{\Rightarrow} \frac{f_1}{f_2} \in M_0(\Gamma) \overset{(a)}{\Rightarrow} f_1 = cf_2$, if we set $f_2 = E_k \in M_k(\Gamma)$ the statement (c) follows.

(d) $f \in S_k(\Gamma) \ \Rightarrow\ v_\infty(f) \geq 1 \ \Rightarrow\ k \geq 12$
$\Delta \in S_{12}(\Gamma) \ \Rightarrow\ v_\infty(\Delta) = 1 \ \wedge\ v_p(\Delta) = 0$ for $p \neq \infty \overset{(v_\infty(f) \geq v_\infty(\Delta))}{\Rightarrow} \frac{f}{\Delta} \in M_{k-12} \ \Rightarrow\ k \neq 14 \ \wedge\ f \in \Delta M_{k-12}(\Gamma)$

(e) For $k > 2 : E_k(\infty) = 1$ (E_k does not vanish at ∞, $v_\infty(E_k) = 0$) $\Rightarrow\ \forall f \in M_k(\Gamma) :\ \exists! c \in \mathbb{C}$ ($c = -f(\infty)$) $:\ f + cE_k$ has $v_\infty(f + cE_k) > 0$, ie. $f + cE_k \in S_k(\gamma) \Rightarrow\ f \in S_k \oplus \mathbb{C}E_k$

$\square$

4.2 Proposition. $\mathcal{B}_k := \{E_4^i E_6^j | i, j \geq 0,\ 4i + 6j = k\}$ *is a basis of* $M_k(\Gamma)$.
The dimension of $M_k(\Gamma)$ *is* $\begin{cases} \lfloor \frac{k}{12} \rfloor, & k \equiv 2 \ mod\ 12 \\ \lfloor \frac{k}{12} \rfloor + 1, & otherwise \end{cases}$.

Proof. First note that the statement is trivial for $k < 0$, $k = 2$ or k odd. We first proof that $M_k(\Gamma)$ is generated by $\mathcal{B}_k$ with the desired dimension, using complete induction on the weight k for both. The linear independence can be proofed by contradiction or by a combinatorial argument, namely that the size of $\mathcal{B}_k \ \Rightarrow\ \mathcal{B}_k$ is equal to the expression for the dimension that was proofed earlier. So $\mathcal{B}_k$ is generating and linearly independent and therefore a basis.

1. *Induction base step*: Prop. 3.1 (a,c) $\Rightarrow$ for $k = 0, 4, 6, 8, 10, 14$: $M_k(\Gamma)$ is 1 dimensional and has the basis 1, E_4, E_6, E_4^2, E_4E_6, $E_4^2E_6$.
2. *Induction step ($\mathcal{B}_k$ generating system)*:
   (a) We can assume $k \geq 12 \Rightarrow \exists i, j \geq 0 : 4i + 6j = k \Rightarrow E_4^i E_6^j \in M_k(\Gamma)$.
   Now let $f \in M_k(\Gamma) \overset{\text{as in (3.1)(e)}}{\Rightarrow} \exists! c_{ij} \in \mathbb{C}$: $f - c_{ij}E_4^iE_6^j \in S_k(\Gamma) \overset{(3.1)(d)}{\Rightarrow} \exists! f_1 \in M_{k-12}$: $f = c_{ij}E_4^iE_6^j + \Delta f_1$
   (b) $\Delta = \frac{(2\pi)^{12}}{1728}\left(E_4^3 - E_6^2\right)$
   (c) By the induction hypothesis we have that $f_1 \in M_{k-12}(\Gamma)$ is generated by $\mathcal{B}_{k-12}$.
   (d) Using (a), (b) and (c) we get that $M_k(\Gamma)$ is generated by $\mathcal{B}_k$.
3. *Induction step (dimension of $M_k(\Gamma)$)*: Using the result (a) above and the induction hypothesis for f_1 we get:

$$Dim_{\mathbb{C}}(M_k(\Gamma)) = 1 + Dim_{\mathbb{C}}(M_{k-12}(\Gamma)) = \begin{cases} 1 + \left\lfloor \frac{k-12}{12} \right\rfloor = \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \ mod\ 12 \\ 1 + \left\lfloor \frac{k-12}{12} \right\rfloor + 1 = \left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{otherwise} \end{cases}$$

4. *Linear independence (contradiction)*: We assume that $\mathcal{B}_k$ is linearly dependent ($k \geq 12$). Therefore we find a linear combination of $E_4^iE_6^j$ with at least two nontrivial coefficients c_{ij} that is zero $\Leftrightarrow$ $\exists m, n \geq 0 : 4m + 6n = k$ (we choose m minimal), st.: $\sum_{i,j\geq 0:4i+6j=k} c_{ij}E_4^iE_6^j = 0,\ c_{mn} \neq 0 \ \Rightarrow$

$$\begin{aligned} 0 \neq -c_{mn} &= \sum_{m\neq i, n\neq j \geq 0:4i+6j=k} c_{ij}\frac{E_4^{(i-m)}}{E_6^{(n-j)}} = \sum_{m\neq i, n\neq j\geq 0:4i+6j=k} c_{ij}\frac{(E_4^3)^{(4i-4m)/12}}{(E_6^2)^{(6n-6j)/12}} \\ &= \sum_{m\neq i,n\neq j\geq 0:4i+6j=k} c_{ij}\left(\frac{E_4^3}{E_6^2}\right)^{(4i-4m)/12} \end{aligned}$$

   Because there is at least one other non trivial coefficient, we get a non trivial polynomial expression in $\frac{E_4^3}{E_6^2}$ with complex coefficients $\Rightarrow$ $\frac{E_4^3}{E_6^2}$ = const. But $v_w(E_4^3) = 3 \neq 0 = v_w(E_6^2)$ Contradiction!.
5. *Linear independence (combinatorial)*: There is an alternative (purely combinatorial) proof for the linear independence using the following lemma:

   4.1 Lemma. *For $k \geq 0$, k even:*

$$\#\left\{i, i \geq 0 | 4i + 6j = k\right\} = \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \ mod\ 12 \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & otherwise \end{cases}$$

   Proof (Lemma). We simplify the equation by dividing by 2. $k \equiv_6 l \in \{0, 1, 2, 3, 4, 5\}$
   $\Rightarrow \left\lfloor \frac{k}{6} \right\rfloor = \frac{k-l}{6} =: m,\ k = 6m + l$

   We need to show: $\#M = \{i, j \geq 0 | 2i + 3j = 6m + l\} = \begin{cases} m, & l = 1 \\ m + 1, & l \neq 1 \end{cases}$

   Generally we have: $2i = 6m + l - 3j \overset{!}{\geq} 0 \ \Rightarrow$

$$\left(0 \leq j \leq 2m + \frac{l}{3}\right) \wedge (j \text{ even} \Leftrightarrow l \text{ even}) \tag{7}$$

   For each such j we find exactly one element in M, so we simply need to count all j's satisfying the condition above. And indeed:
   For $l = 1 : 0 \leq j \leq 2m + \frac{1}{3} \ \wedge\ j$ odd $\Rightarrow$ we have m possible choices for j.
   For $l \neq 1$ we have $m + 1$ possible choices for j as is easily checked (the same way as for $l = 1$). $\square$

Applying the lemma we get:

$$\#\{f \in \mathcal{B}_k\} = \#\{i, i \geq 0 | 4i + 6j = k\}$$
$$= \begin{cases} \left\lfloor \frac{k}{12} \right\rfloor, & k \equiv 2 \ mod \ 12 \\ \left\lfloor \frac{k}{12} \right\rfloor + 1, & \text{otherwise} \end{cases} = Dim_{\mathbb{C}}(M_k(\Gamma))$$

And we get that $\mathcal{B}_k$ is linearly independent.

□

4.3 Proposition (Corollary)**.** *$\mathcal{A} := \bigoplus_{k=0}^{\infty} M_k(\Gamma)$ defines a (graded) $\mathbb{C}$-algebra and $\mathcal{A} \cong \mathbb{C}[\mathbf{x}, \mathbf{y}]$.*

Proof. That $\mathcal{A}$ is an algebra is easily checked and by using Prop. 4.2 we immediately get that the following map (extended to $\mathbb{C}[x, y]$) defines an algebra isomorphism:

$$\varphi : \mathbb{C}[x, y] \xrightarrow{\sim} \mathcal{A}$$
$$x \mapsto E_4, \ y \mapsto E_6$$

□