PDE characterization of Stochastic Target Problems with Controlled Loss in Jump Diffusion Models

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For $0 \leq t \leq T$, we are given two controlled diffusion processes $\left\{X^{\nu}_{t,x}(s), t \leq s \leq T\right\}$ and $\left\{Y^{\nu}_{t,x,y}(s), t \leq s \leq T\right\}$ with values respectively in $\mathbb{R}^d$ and $\mathbb{R}$, starting at time t in (x, y). The aim of the controller is to find the minimal initial condition y for which it is possible to find a control ν satisfying $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p$ for some given measurable map Ψ, non-decreasing in the y-variable, and for a level p. Namely, he wants to compute:

$$v(t,x,p) := \inf\left\{y \geq -\kappa : \mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\right] \geq p \text{ for some control } \nu\right\}, \tag{1}$$

where $\kappa \in \mathbb{R}_+$. When $p = 1$ and $\Psi(x, y) = 1_{\{V(x,y)\geq 0\}}$ for some map V, then $v(t, x, 1)$ coincides with the stochastic target problem studied in Bouchard (2002), and in Soner and Touzi (2002) for Brownian controlled SDEs. In the above mentioned papers, the authors restricted to the setting of controls with values in a compact subset of $\mathbb{R}^d$. Their proofs are heavily relying on this assumption. One may note that for $V(x, y) = y - g(x)$ for some function g, $v(t, x, 1)$ coincides with the superreplication price of the claim $g\left(X^{\nu}_{t,x}\right)$.

Such problems have already been discussed by Follmer and Leukert (1999) in the context of financial mathematics. However, they used a duality argument which does not extend to general non linear controlled diffusion cases.

In order to deal with the problem 1, Bouchard, Elie and Touzi (2009) introduced an additional controlled diffusion process $P^{\alpha}_{t,p}$, which appears to (essentially) correspond to the conditional probability of reaching the target $V\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right) \geq 0$. This allowed them to reformulate the problem and to rewrite the problem into a classical stochastic target problem already studied by Soner and Touzi. However, the new control α appearing in the diffusion P^{α} can no longer be assumed to take values in a compact set, as it is given by the martingale representation theorem.

The aim of this presentation is to extend the work of Bouchard, Elie and Touzi (2009) to the setting of jump diffusions, in order to obtain a PDE characterization for the value function 1. This approach allows us to deal with general problem, such as the utility indifference pricing, quantile hedging, loss functions among others.

We follow the key idea of Bouchard, Elie and Touzi (2009) so as to convert the problem v into a singular stochastic target problem. This is done by diffusing the conditional expectation $\mathbb{E}\left[\Psi\left(X^{\nu}_{t,x}(T), Y^{\nu}_{t,x,y}(T)\right)\middle|\mathcal{F}_s\right]$ for $s \in [t, T]$, and considering it as an additional controlled state variable $P^{\alpha,\chi}_{t,p}$, where the additional control χ comes from the jump part of the martingale representation.

The main new technical difficulty is due to the presence of jumps and of the new control χ. First, it leads to an additional (non-local) term in the PDE characterization. Moreover, part of the control now takes values in an unbounded set of measurable maps. This also leads to a new (non-trivial) relaxation of the non-local part of the associated operator, in comparison to Bouchard, Elie and Touzi (2009). We shall see that the convex face-lifting phenomenon in the p-variable observed by Bouchard, Elie and Touzi (2009) at the terminal date extends to a much more general context.