

B. Standard Statistical Methods for Market Risk

1. Variance-Covariance Method
2. Historical Simulation Method
3. Monte Carlo Simulation Method
4. An Example
5. Improving the Statistical Toolkit

B1. Variance-Covariance Method

Further Assumptions

- We assume \mathbf{X}_{t+1} has a *multivariate normal* distribution (either unconditionally or conditionally).
- We assume that the linearized loss in terms of risk factors is a sufficiently *accurate approximation* of the loss. We consider the problem of estimating the distribution of

$$L^\Delta = l_{[t]}^\Delta(\mathbf{X}_{t+1}),$$

Theory Behind Method

Assume $\mathbf{X}_{t+1} \sim N_d(\boldsymbol{\mu}, \Sigma)$.

Assume the linearized loss operator (??) has been determined and write this for convenience as

$$l_{[t]}^{\Delta}(\mathbf{x}) = - \left(c + \sum_{i=1}^d w_i x_i \right) = -(c + \mathbf{w}'\mathbf{x}).$$

The loss distribution is approximated by the distribution of $L^{\Delta} = l_{[t]}^{\Delta}(\mathbf{X}_{t+1})$.

Now since $\mathbf{X}_{t+1} \sim N_d(\boldsymbol{\mu}, \Sigma) \Rightarrow \mathbf{w}'\mathbf{X}_{t+1} \sim N(\mathbf{w}'\boldsymbol{\mu}, \mathbf{w}'\Sigma\mathbf{w})$, we have

$$L^{\Delta} \sim N(-c - \mathbf{w}'\boldsymbol{\mu}, \mathbf{w}'\Sigma\mathbf{w}).$$

Implementing the Method

1. The constant terms in c and w are calculated
2. The *mean vector* μ and *covariance matrix* Σ are estimated from data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$ to give estimates $\hat{\mu}$ and $\hat{\Sigma}$.
3. Inference about the loss distribution is made using distribution $N(-c - \mathbf{w}'\hat{\mu}, \mathbf{w}'\hat{\Sigma}\mathbf{w})$
4. Estimates of the risk measures VaR_α and ES_α are calculated from the estimated distribution of L^Δ .

Estimating Risk Measures

- Value-at-Risk. VaR_α is estimated by

$$\widehat{\text{VaR}}_\alpha = -c - \mathbf{w}'\widehat{\boldsymbol{\mu}} + \sqrt{\mathbf{w}'\widehat{\boldsymbol{\Sigma}}\mathbf{w}} \cdot \Phi^{-1}(\alpha).$$

- Expected Shortfall. ES_α is estimated by

$$\widehat{ES}_\alpha = -c - \mathbf{w}'\widehat{\boldsymbol{\mu}} + \sqrt{\mathbf{w}'\widehat{\boldsymbol{\Sigma}}\mathbf{w}} \cdot \frac{\phi(\Phi^{-1}(\alpha))}{1 - \alpha}.$$

Remark. For a rv $Y \sim N(0, 1)$ it can be shown that $E(Y \mid Y > \Phi^{-1}(\alpha)) = \phi(\Phi^{-1}(\alpha))/(1 - \alpha)$ where ϕ is standard normal density and Φ the df.

Pros and Cons, Extensions

- Pros. In contrast to the methods that follow, variance-covariance offers *analytical solution* with no simulation.
- Cons. Linearization may be crude approximation. Assumption of normality may seriously *underestimate tail* of loss distribution.
- Extensions. Instead of assuming normal risk factors, the method could be easily adapted to use multivariate Student t risk factors or multivariate hyperbolic risk factors, without sacrificing tractability. (Method works for all elliptical distributions.)

B2. Historical Simulation Method

The Idea

Instead of estimating the distribution of $L = l_{[t]}(\mathbf{X}_{t+1})$ under some explicit parametric model for \mathbf{X}_{t+1} , estimate distribution of the loss operator under *empirical distribution* of data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$.

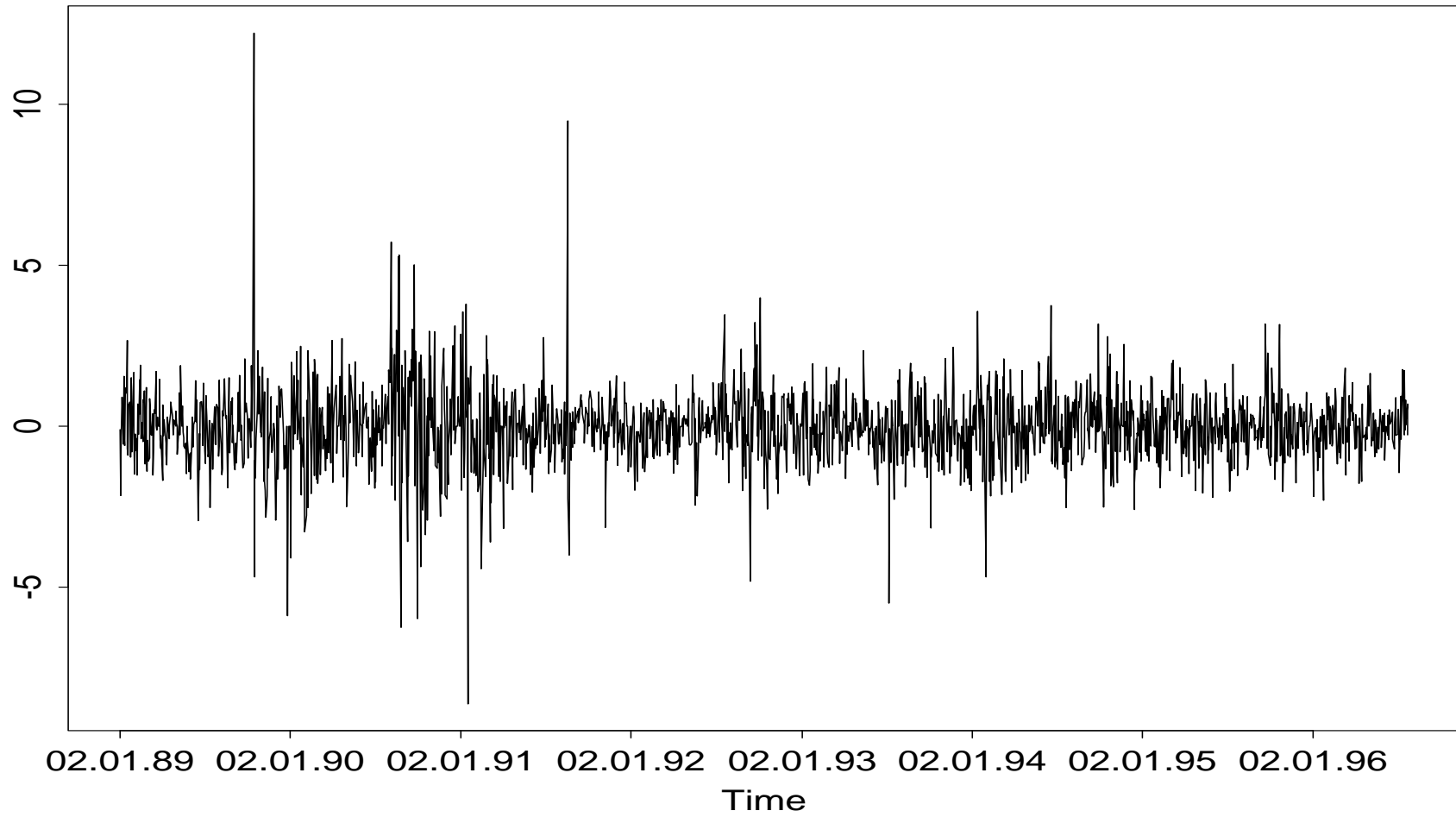
The Method

1. Construct the *historical simulation data*

$$\{\tilde{L}_s = l_{[t]}(\mathbf{X}_s) : s = t - n + 1, \dots, t\} \quad (1)$$

2. Make inference about loss distribution and risk measures using these historically simulated data: $\tilde{L}_{t-n+1}, \dots, \tilde{L}_t$.

Historical Simulation Data: Percentage Losses



Inference about loss distribution

There are various possibilities in a simulation approach:

- Use *empirical quantile estimation* to estimate the VaR directly from the simulated data. But what about precision?
- Fit a parametric univariate distribution to $\tilde{L}_{t-n+1}, \dots, \tilde{L}_t$ and calculate risk measures from this distribution.
But which distribution, and will it model the *tail*?
- Use the techniques of *extreme value theory* to estimate the tail of the loss distribution and related risk measures.

Theoretical Justification

If $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$ are iid or more generally stationary, convergence of empirical distribution to true distribution is ensured by suitable version of law of large numbers.

Pros and Cons

- Pros. Easy to implement. No statistical estimation of the distribution of \mathbf{X} necessary.
- Cons. It may be difficult to collect sufficient quantities of relevant, synchronized data for all risk factors. Historical data may not contain examples of extreme scenarios.

B3. The Monte Carlo Method

Idea

We estimate the distribution of $L = l_{[t]}(\mathbf{X}_{t+1})$ under some explicit parametric model for \mathbf{X}_{t+1} .

In contrast to the variance-covariance approach we do not necessarily make the problem analytically tractable by linearizing the loss and making an assumption of normality for the risk factors.

Instead we make inference about L using Monte Carlo methods, which involves *simulation* of new risk factor data.

The Method

1. With the help of the historical risk factor data $\mathbf{X}_{t-n+1}, \dots, \mathbf{X}_t$ calibrate a suitable statistical model for risk factor changes and simulate m new data $\tilde{\mathbf{X}}_{t+1}^{(1)}, \dots, \tilde{\mathbf{X}}_{t+1}^{(m)}$ from this model.
2. Construct the Monte Carlo data $\{\tilde{L}_i = l_{[t]}(\tilde{\mathbf{X}}_{t+i}^{(i)}), i = 1, \dots, m\}$.
3. Make inference about loss distribution and risk measures using the simulated data $\tilde{L}_1, \dots, \tilde{L}_m$. We have similar possibilities as for historical simulation.

Pros and Cons

- Pros. Very general. No restriction in our choice of distribution for \mathbf{X}_{t+1} .
- Cons. Can be very time consuming if loss operator is difficult to evaluate, which depends on size and complexity of portfolio.

Note that MC approach does not address the problem of determining the distribution of \mathbf{X}_{t+1} .

B4. An Example With BMW-SIEMENS Data

```
> Xdata <- DAX[(5147:6146),c("BMW","SIEMENS")]
> X <- seriesData(Xdata)

# Set stock prices and number of units
> alpha <- cbind(1,10)
> Sprice <- cbind(844,76.9)

#1. Implement variance-covariance analysis

> weights <- alpha*Sprice
> muhat <- apply(X,2,mean)
> Sigmahat <- var(X)
> meanloss <- -sum(weights*muhat)
> varloss <- weights %*% Sigmahat %*% t(weights)
> VaR99 <- meanloss + sqrt(varloss)*qnorm(0.99)
> ES99 <- meanloss +sqrt(varloss)*dnorm(qnorm(0.99))/0.01

#2. Implement a historical simulation analysis

> loss.operator <- function(x,weights){
-apply((exp(x)-1)*matrix(weights,nrow=dim(x)[1],ncol=length(weights),byrow=T),1,sum)}
> hsdata <- loss.operator(X,weights)
> VaR99.hs <- quantile(hsdata,0.99)
> ES99.hs <- mean(hsdata[hsdata > VaR99.hs])
```

Example Continued

#3a. Implement a Monte Carlo simulation analysis with Gaussian risk factors

```
> X.new <- rmnorm(10000,Sigma=Sigmahat,mu=muhat)
> mcdata <- loss.operator(X.new,weights)
> VaR99.mc <- quantile(mcdata,0.99)
> ES99.mc <- mean(mcdata[mcdata > VaR99.mc])
```

#3b. Implement alternative Monte Carlo simulation analysis with t risk factors

```
> model <- fit.t(X, nu=NA)
> X.new <- rmt(10000, df=model$nu, Sigma=model$Sigma, mu=model$mu)
> mcdataat <- loss.operator(X.new,weights)
> VaR99.mct <- quantile(mcdataat,0.99)
> ES99.mct <- mean(mcdataat[mcdataat > VaR99.mct])
```

#Draw pictures

```
> hist(hsdata,nclass=20,prob=T)
> abline(v=c(VaR99,ES99))
> abline(v=c(VaR99.hs,ES99.hs),col=2)
> abline(v=c(VaR99.mc,ES99.mc),col=3)
> abline(v=c(VaR99.mct,ES99.mct),col=4)
```

Software for the example

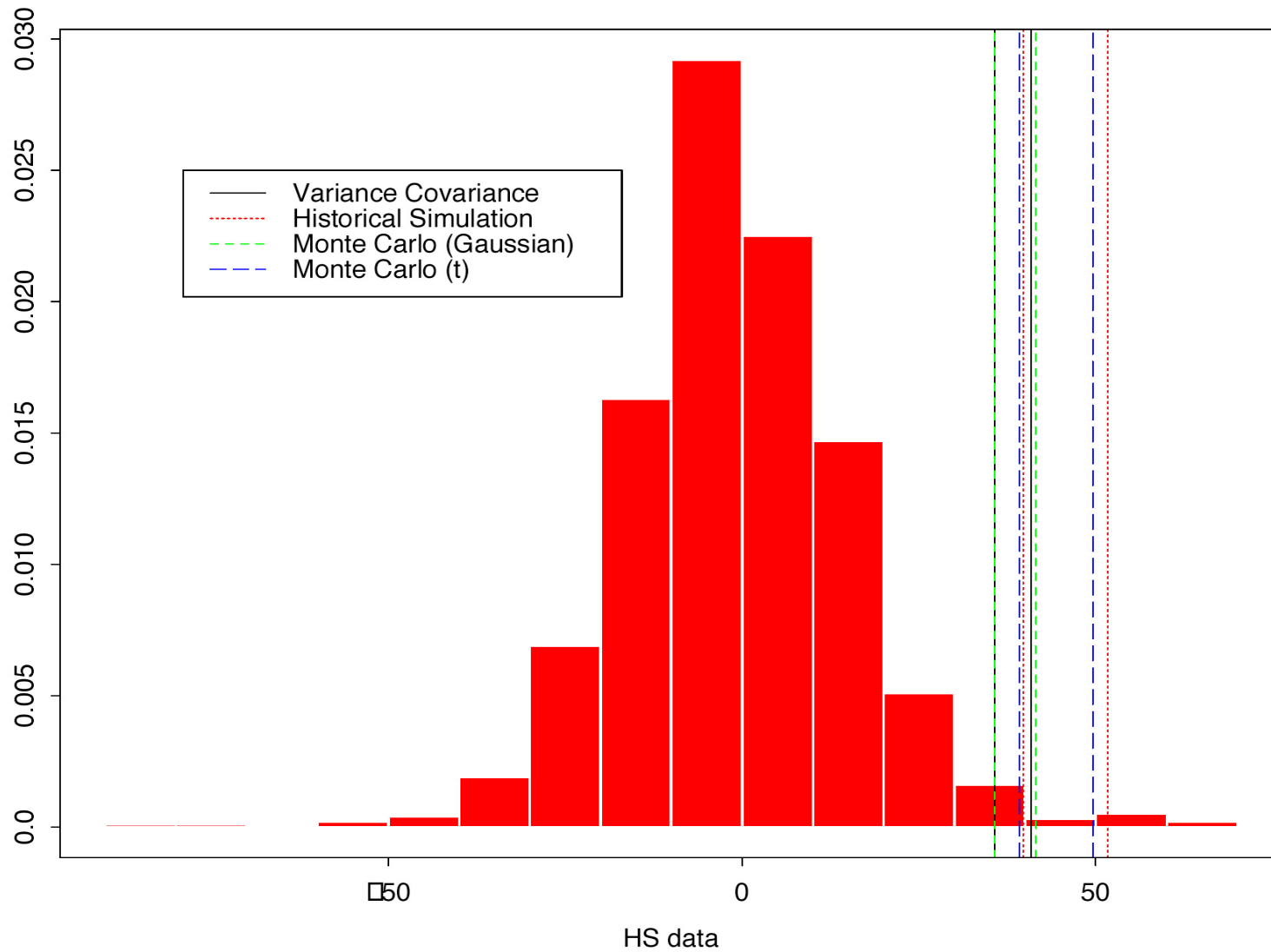
The above code is for the software package S-PLUS.

You can also use the software package R, which is free. The code for R is very similar.

For both S-PLUS and R, you need the QRMLib package; see Alexander McNeil's website

<http://www.ma.hw.ac.uk/mcneil/book/QRMLib.html>

Comparison of Risk Measure Estimates



B5. Improving the Statistical Toolkit

Questions we will examine in the remainder of this course include the following.

Multivariate Models

Are there alternatives to the multivariate normal distribution for modelling changes in several risk factors?

We will expand our stock of multivariate models to include multivariate *normal mixture* models and *copula models*. These will allow a more realistic description of joint extreme risk factor changes.

Improving the Statistical Toolkit II

Monte Carlo Techniques

How can we simulate dependent risk factor changes?

We will look in particular at ways of *simulating multivariate risk factors* in non-Gaussian models.

We will not examine the following:

Conditional Risk Measurement

How can we implement a genuinely conditional calculation of risk measures that takes the dynamics of risk factors into consideration?

We can consider methodology for modelling financial *time series* and predicting volatility, particularly using *GARCH* models.

References

On risk management:

- [Crouhy et al., 2010]
- [Jorion, 2007]

Bibliography

- [Crouhy et al., 2010] Crouhy, M., Galai, D., and Mark, R. (2010). *Risk Management*. McGraw-Hill, New York, 2nd edition.
- [Jorion, 2007] Jorion, P. (2007). *Value at Risk: The New Benchmark for Measuring Financial Risk*. McGraw-Hill, New York, 3rd edition.