D. Normal Mixture Models and Elliptical Models

1. Normal Variance Mixtures

2. Normal Mean-Variance Mixtures

3. Spherical Distributions

4. Elliptical Distributions

D1. Multivariate Normal Mixture Distributions

Pros of Multivariate Normal Distribution

- inference is "well known" and estimation is "easy".
- distribution is given by $\boldsymbol{\mu}$ and Σ.
- linear combinations are normal ($\rightarrow$ VaR and ES calcs easy).
- conditional distributions are normal.
- For $(X_1, X_2)^\top \sim N_2(\boldsymbol{\mu}, \Sigma)$,

$$\rho(X_1, X_2) = 0 \quad \Longleftrightarrow \quad X_1 \text{ and } X_2 \text{ are independent.}$$

Multivariate Normal Variance Mixtures

Cons of Multivariate Normal Distribution

- tails are thin, meaning that extreme values are scarce in the normal model.
- joint extremes in the multivariate model are also too scarce.
- the distribution has a strong form of symmetry, called elliptical symmetry.

How to repair the drawbacks of the multivariate normal model?

Multivariate Normal Variance Mixtures

The random vector $\mathbf{X}$ has a (multivariate) normal variance mixture distribution if

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \boldsymbol{\mu} + \sqrt{W} A \mathbf{Z}, \tag{1}$$

where

- $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$;
- $W \geq 0$ is a scalar random variable which is independent of $\mathbf{Z}$; and
- $A \in \mathbb{R}^{d \times k}$ and $\boldsymbol{\mu} \in \mathbb{R}^d$ are a matrix and a vector of constants, respectively.

Set $\Sigma := AA^\top$. Observe: $\mathbf{X} | W = w \sim N_d(\boldsymbol{\mu}, w\Sigma)$.

Multivariate Normal Variance Mixtures

Assumption: rank(A)$= d \leq k$, so Σ is a positive definite matrix.

If $E(W) < \infty$ then easy calculations give

$$E(\mathbf{X}) = \boldsymbol{\mu} \quad \text{and} \quad \text{cov}(\mathbf{X}) = E(W)\Sigma.$$

We call $\boldsymbol{\mu}$ the *location vector* or *mean vector* and we call Σ the *dispersion matrix*.

The correlation matrices of $\mathbf{X}$ and $A\mathbf{Z}$ are identical:

$$\text{corr}(\mathbf{X}) = \text{corr}(A\mathbf{Z}).$$

Multivariate normal variance mixtures provide the most useful examples of *elliptical* distributions.

Properties of Multivariate Normal Variance Mixtures

1. Characteristic function of multivariate normal variance mixtures

$$\begin{aligned}\phi_{\mathbf{X}}(\mathbf{t}) &= E\left(\exp\{i\mathbf{t}^\top \mathbf{X}\}\right) \\ &= E\left(E\left(\exp\{i\mathbf{t}^\top \mathbf{X}\}|W\right)\right) \\ &= E\left(\exp\{i\mathbf{t}^\top \boldsymbol{\mu} - \frac{1}{2}W\mathbf{t}^\top \Sigma \mathbf{t}\}\right).\end{aligned}$$

Denote by H the d.f. of W. Define the Laplace-Stieltjes transform of H

$$\hat{H}(\theta) := E(e^{-\theta W}) = \int_0^\infty e^{-\theta u} dH(u).$$

Then

$$\phi_{\mathbf{X}}(\mathbf{t}) = \exp\{i\mathbf{t}^\top \boldsymbol{\mu}\}\hat{H}\left(\frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}\right).$$

Based on this, we use the notation $\mathbf{X} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{H})$.

Properties of Multivariate Normal Variance Mixtures

2. Linear operations. For $\mathbf{X} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{H})$ and $\mathbf{Y} = B\mathbf{X} + \mathbf{b}$, where $B \in \mathbb{R}^{k \times d}$ and $\mathbf{b} \in \mathbb{R}^k$, we have

$$\mathbf{Y} \sim M_k(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B^\top, \hat{H}).$$

As a special case, if $\mathbf{a} \in \mathbb{R}^d$,

$$\mathbf{a}^\top \mathbf{X} \sim M_1(\mathbf{a}^\top \boldsymbol{\mu}, \mathbf{a}^\top \Sigma \mathbf{a}, \hat{H}).$$

Proof:

$$\begin{aligned}\phi_{\mathbf{Y}}(\mathbf{t}) = E\left(e^{i\mathbf{t}^\top (B\mathbf{X}+\mathbf{b})}\right) &= e^{i\mathbf{t}^\top \mathbf{b}}\, \phi_{\mathbf{X}}(B^\top \mathbf{t}) \\ &= e^{i\mathbf{t}^\top (\mathbf{b}+B\boldsymbol{\mu})}\, \hat{H}\left(\frac{1}{2}\mathbf{t}^\top B\Sigma B^\top \mathbf{t}\right).\end{aligned}$$

Properties of Multivariate Normal Variance Mixtures

3. Density. If $P[W = 0] = 0$ then as $\mathbf{X}|W = w \sim N_d(\boldsymbol{\mu}, w\Sigma)$,

$$f_{\mathbf{X}}(\mathbf{x}) = \int_0^\infty f_{\mathbf{X}|W}(\mathbf{x}|w)dH(w)$$
$$= \int_0^\infty \frac{w^{-d/2}}{(2\pi)^{d/2}|\Sigma|^{1/2}} \exp\left\{-\frac{(\mathbf{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}{2w}\right\} dH(w).$$

The density depends on $\mathbf{x}$ only through $(\mathbf{x}-\boldsymbol{\mu})^\top\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})$.

Properties of Multivariate Normal Variance Mixtures

4. Independence.

If Σ is diagonal, then the components of $\mathbf{X}$ are *uncorrelated*.

But, in general, they are not independent,

e.g. for $\mathbf{X} \sim M_2(\boldsymbol{\mu}, I_2, \hat{H})$,

$$\rho(X_1, X_2) = 0 \not\Rightarrow X_1 \text{ and } X_2 \text{ are independent.}$$

Indeed, X_1 and X_2 are independent iff W is a.s. constant.

i.e. when $\mathbf{X} = (X_1, X_2)^\top$ is multivariate normally distributed.

Examples of Multivariate Normal Variance Mixtures

Two point mixture

$$W = \begin{cases} k_1 \text{ with probability } p, \\ k_2 \text{ with probability } 1-p \end{cases} \qquad k_1, k_2 > 0, k_1 \neq k_2.$$

Could be used to model two regimes - ordinary and stress.

Multivariate t

W has an inverse gamma distribution, $W \sim \mathrm{Ig}(\nu/2, \nu/2)$.

Equivalently, $\frac{\nu}{W} \sim \chi^2_\nu$.

This gives multivariate t with ν degrees of freedom.

Symmetric generalised hyperbolic

W has a GIG (generalised inverse Gaussian) distribution.

The Multivariate t Distribution

Density of multivariate t

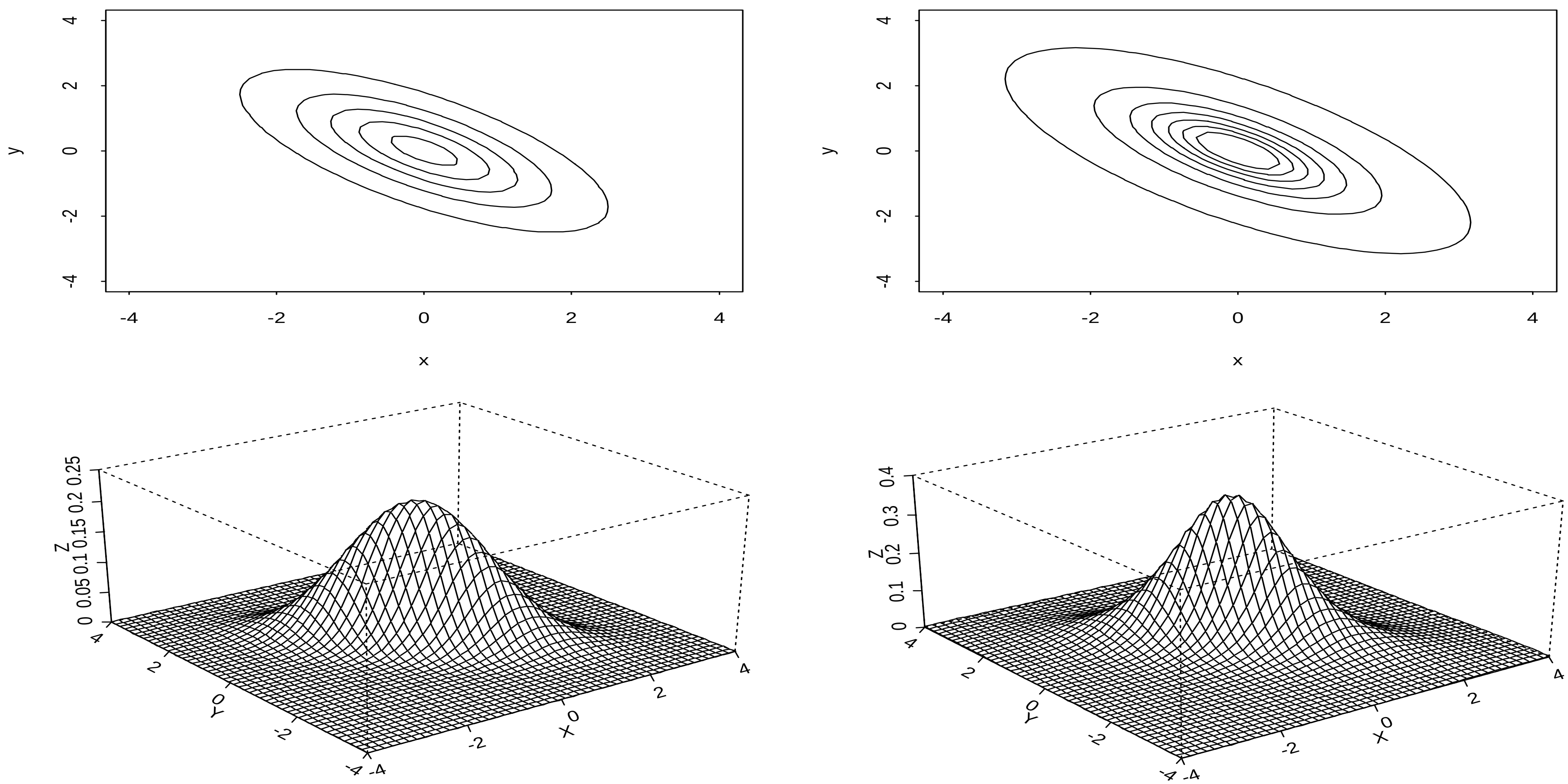
$$f(\mathbf{x}) = k_{\Sigma,\nu,d}\left(1 + \frac{(\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}{\nu}\right)^{-\frac{(\nu+d)}{2}}$$

where $\boldsymbol{\mu} \in \mathbb{R}^d$, $\Sigma \in \mathbb{R}^{d\times d}$ is a positive definite matrix, ν is the degrees of freedom and $k_{\Sigma,\nu,d}$ is a normalizing constant.

- $E(\mathbf{X}) = \boldsymbol{\mu}$.
- As $E(W) = \frac{\nu}{\nu-2}$, we get $\operatorname{cov}(\mathbf{X}) = \frac{\nu}{\nu-2}\Sigma$. For finite variances/correlations, $\nu > 2$.

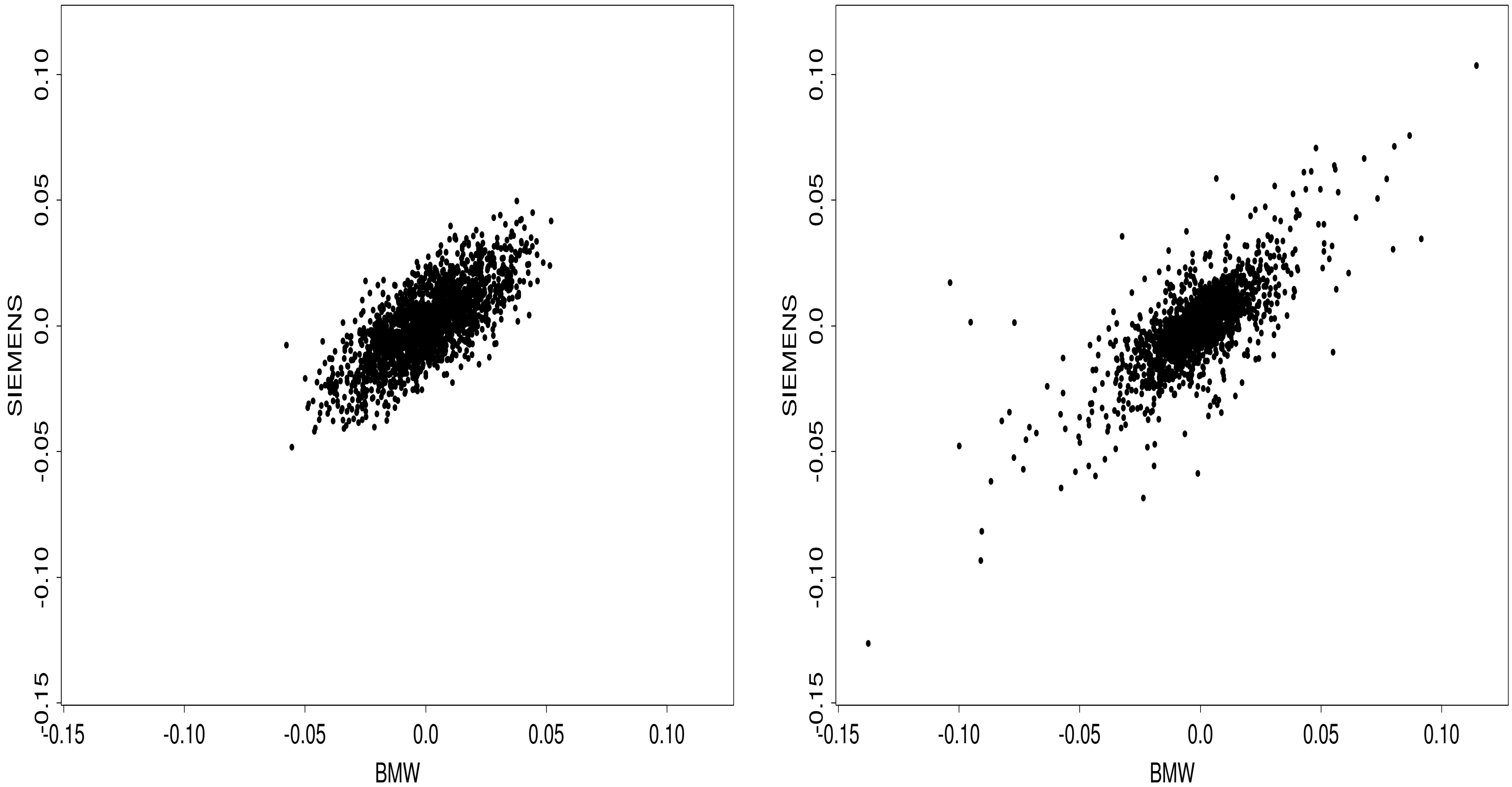
Notation: $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$.

Bivariate Normal and t



Left plot is bivariate normal, right plot is bivariate t with $\nu = 3$. Mean is zero, all variances equal 1 and $\rho = -0.7$.

Fitted Normal and t_3 Distributions



Simulated data (2000) from models fitted by maximum likelihood to BMW-Siemens data. Left plot is fitted normal, right plot is fitted t_3.

Simulating Normal Variance Mixture Distributions

To simulate $\mathbf{X} \sim M_d(\boldsymbol{\mu}, \Sigma, \hat{H})$.

1. Generate $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$, with $\Sigma = AA^\top$.

2. Generate W with df H (with Laplace-Stieltjes transform $\hat{H}$), independent of $\mathbf{Z}$.

3. Set $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W} A\mathbf{Z}$.

Simulating Normal Variance Mixture Distributions

Example: t **distribution**

To simulate a vector $\mathbf{X} \sim t_d(\nu, \boldsymbol{\mu}, \Sigma)$.

1. Generate $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$, with $\Sigma = AA^\top$.

2. Generate $V \sim \chi^2_\nu$ and set $W = \frac{\nu}{V}$.

3. Set $\mathbf{X} = \boldsymbol{\mu} + \sqrt{W} A\mathbf{Z}$.

Symmetry in Normal Variance Mixture Distributions

Elliptical symmetry means 1-dimensional margins are symmetric.

Observation for stock returns: negative returns (losses) have heavier tails than positive returns (gains).

Introduce asymmetry by mixing normal distributions with different means as well as different variances.

This gives the class of multivariate normal mean-variance mixtures.

D2. Multivariate Normal Mean-Variance Mixtures

The random vector $\mathbf{X}$ has a (multivariate) normal mean-variance mixture distribution if

$$\mathbf{X} \stackrel{\mathrm{d}}{=} \mathbf{m}(W) + \sqrt{W} A \mathbf{Z}, \tag{2}$$

where

- $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$;
- $W \geq 0$ is a scalar random variable which is independent of $\mathbf{Z}$; and
- $A \in \mathbb{R}^{d \times k}$ and $\boldsymbol{\mu} \in \mathbb{R}^d$ are a matrix and a vector of constants, respectively.
- $\mathbf{m} : [0, \infty) \to \mathbb{R}^d$ is a measurable function.

Normal Mean-Variance Mixtures

Normal mean-variance mixture distributions add asymmetry.

In general, they are *no longer elliptical* and $\mathrm{corr}(\mathbf{X}) \neq \mathrm{corr}(A\mathbf{Z})$.

Set $\Sigma := AA^\top$. Observe:

$$\mathbf{X}|W = w \sim N_d(\mathbf{m}(w), w\Sigma).$$

A concrete specification of $\mathbf{m}(W)$ is $\mathbf{m}(W) = \boldsymbol{\mu} + W\boldsymbol{\gamma}$.

Example: Let W have generalized inverse Gaussian distribution to get $\mathbf{X}$ generalised hyperbolic.

$\gamma = \mathbf{0}$ places us back in the (elliptical) normal variance mixture family.

D3. Spherical Distributions

Recall that a map $U \in \mathbb{R}^{d \times d}$ is orthogonal if $UU^\top = U^\top U = I_d$.

A random vector $\mathbf{Y} = (Y_1, \ldots, Y_d)^\top$ has a *spherical* distribution if for every orthogonal map $U \in \mathbb{R}^{d \times d}$

$$\mathbf{Y} \stackrel{\mathsf{d}}{=} U\mathbf{Y}.$$

Use $\|\cdot\|$ to denote the Euclidean norm, i.e. for $\mathbf{t} \in \mathbb{R}^d$, $\|\mathbf{t}\| = (t_1^2 + \cdots + t_d^2)^{1/2}$.

Spherical Distributions

THEOREM The following are equivalent.

1. $\mathbf{Y}$ is spherical.

2. There exists a function ψ of a scalar variable such that

$$\phi_{\mathbf{Y}}(\mathbf{t}) = E(e^{i\mathbf{t}^\top \mathbf{Y}}) = \psi(\|\mathbf{t}\|^2), \quad \forall \mathbf{t} \in \mathbb{R}^d.$$

3. For every $\mathbf{a} \in \mathbb{R}^d$,

$$\mathbf{a}^\top \mathbf{Y} \stackrel{\mathsf{d}}{=} \|\mathbf{a}\| Y_1.$$

We call ψ the characteristic generator of the spherical distribution.

Notation: $\mathbf{Y} \sim S_d(\psi)$.

Examples of Spherical Distributions

- $\mathbf{X} \sim N_d(\mathbf{0}, I_d)$ is spherical. The characteristic function is

$$\phi_{\mathbf{X}}(\mathbf{t}) = E(e^{i\mathbf{t}^\top \mathbf{X}}) = \exp\left(-\frac{1}{2}\mathbf{t}^\top \mathbf{t}\right).$$

Then $\mathbf{X} \sim S_d(\psi)$ with $\psi(t) = \exp\left(-\frac{1}{2}t\right)$.

- $\mathbf{X} \sim M_d(\mathbf{0}, I_d, \hat{H})$ is spherical, i.e. $\mathbf{X} \stackrel{\mathrm{d}}{=} \sqrt{W}\mathbf{Z}$.

The characteristic function is

$$\phi_{\mathbf{X}}(\mathbf{t}) = \hat{H}\left(\frac{1}{2}\mathbf{t}^\top \mathbf{t}\right).$$

Then $\mathbf{X} \sim S_d(\psi)$ with $\psi(t) = \hat{H}(\frac{1}{2}t)$.

D4. Elliptical distributions

A random vector $\mathbf{X} = (X_1, \ldots, X_d)^\top$ is called *elliptical* if it is an affine transform of a spherical random vector $\mathbf{Y} = (Y_1, \ldots, Y_k)^\top$, i.e.

$$\mathbf{X} \stackrel{\mathsf{d}}{=} \boldsymbol{\mu} + A\mathbf{Y},$$

where $\mathbf{Y} \sim S_k(\psi)$ and $A \in \mathbb{R}^{d \times k}$, $\boldsymbol{\mu} \in \mathbb{R}^d$ are a matrix and vector of constants, respectively.

Set $\Sigma = AA^\top$.

Example: Multivariate normal variance mixture distributions $\mathbf{X} \stackrel{\mathsf{d}}{=} \boldsymbol{\mu} + \sqrt{W} A\mathbf{Z}$.

Properties of Elliptical Distributions

1. Characteristic function of elliptical distributions

The characteristic function is

$$\phi_{\mathbf{X}}(\mathbf{t}) = E(e^{i\mathbf{t}^\top \mathbf{X}}) = E(e^{i\mathbf{t}^\top (\boldsymbol{\mu} + A\mathbf{Y})}) = e^{i\mathbf{t}^\top \boldsymbol{\mu}} \psi\left(\mathbf{t}^\top \Sigma \mathbf{t}\right)$$

Notation: $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$.

We call $\boldsymbol{\mu}$ the location vector, Σ the dispersion matrix and ψ the characteristic generator.

Remark: $\boldsymbol{\mu}$ is unique but Σ and ψ are only unique up to a positive constant, since for any $c > 0$,

$$\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi) \sim E_d\left(\boldsymbol{\mu}, c\Sigma, \psi\left(\frac{\cdot}{c}\right)\right)$$

Properties of Elliptical Distributions

2. Linear operations. For $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$ and $\mathbf{Y} = B\mathbf{X} + \mathbf{b}$, where $B \in \mathbb{R}^{k \times d}$ and $\mathbf{b} \in \mathbb{R}^k$, we have

$$\mathbf{Y} \sim E_k(B\boldsymbol{\mu} + \mathbf{b}, B\Sigma B^\top, \psi).$$

As a special case, if $\mathbf{a} \in \mathbb{R}^d$,

$$\mathbf{a}^\top \mathbf{X} \sim E_1(\mathbf{a}^\top \boldsymbol{\mu}, \mathbf{a}^\top \Sigma \mathbf{a}, \psi).$$

Proof:

$$\begin{aligned} \phi_{\mathbf{Y}}(\mathbf{t}) = E\left(e^{i\mathbf{t}^\top (B\mathbf{X}+\mathbf{b})}\right) &= e^{i\mathbf{t}^\top \mathbf{b}}\, \phi_{\mathbf{X}}(B^\top \mathbf{t}) \\ &= e^{i\mathbf{t}^\top (\mathbf{b}+B\boldsymbol{\mu})}\, \psi\left(\mathbf{t}^\top B\Sigma B^\top \mathbf{t}\right). \end{aligned}$$

Properties of Elliptical Distributions

3. Marginal distributions. For $\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$, set

$$\mathbf{X}_1 = (X_1, \ldots, X_k)^\top \quad \text{and} \quad \mathbf{X}_2 = (X_{k+1}, \ldots, X_d)^\top$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \quad \text{and} \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

Then

$$\mathbf{X}_1 \sim E_k(\boldsymbol{\mu}_1, \Sigma_{11}, \psi) \qquad \mathbf{X}_2 \sim E_{d-k}(\boldsymbol{\mu}_2, \Sigma_{22}, \psi).$$

Properties of Elliptical Distributions

4. Conditional distributions. The conditional distribution of $\mathbf{X}_2 | \mathbf{X}_1 = \mathbf{x}_1$ is elliptical, but in general with a *different* characteristic generator $\tilde{\psi}$.

In the special case of multivariate normality, the characteristic generator remains the same.

Properties of Elliptical Distributions

5. Convolutions. Let $\mathbf{X}$ and $\mathbf{Y}$ be *independent* and

$$\mathbf{X} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi) \qquad \mathbf{Y} \sim E_d(\tilde{\boldsymbol{\mu}}, \tilde{\Sigma}, \tilde{\psi}).$$

If $\Sigma = \tilde{\Sigma}$ then

$$\mathbf{X} + \mathbf{Y} \sim E_d(\boldsymbol{\mu} + \tilde{\boldsymbol{\mu}}, \Sigma, \bar{\psi}),$$

where $\bar{\psi}(u) := \psi(u)\tilde{\psi}(u)$.

Properties of Elliptical Distributions

- The density of an elliptical distribution is constant on ellipsoids.

- Many of the nice properties of the multivariate normal are preserved. In particular, all linear combinations $a_1 X_1 + \ldots + a_d X_d$ are of the same type.

- All marginal distributions are of the same type.

Two rvs X and Y (or their distributions) are of the *same type* if there exist constants $a > 0$ and $b \in \mathbb{R}$ such that $X \stackrel{\mathsf{d}}{=} aY + b$.

References

- [Barndorff-Nielsen and Shephard, 1998] (generalized hyperbolic distributions)
- [Barndorff-Nielsen, 1997] (NIG distribution)
- [Eberlein and Keller, 1995]) (hyperbolic distributions)
- [Prause, 1999] (GH distributions - PhD thesis)
- [Fang et al., 1990] (elliptical distributions)
- [Embrechts et al., 2002] (elliptical distributions in RM)

Bibliography

[Barndorff-Nielsen, 1997] Barndorff-Nielsen, O. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scand. J. Statist.*, 24:1–13.

[Barndorff-Nielsen and Shephard, 1998] Barndorff-Nielsen, O. and Shephard, N. (1998). Aggregation and model construction for volatility models. Preprint, Center for Analytical Finance, University of Aarhus.

[Eberlein and Keller, 1995] Eberlein, E. and Keller, U. (1995). Hyperbolic distributions in finance. *Bernoulli*, 1:281–299.

[Embrechts et al., 2002] Embrechts, P., McNeil, A., and Straumann, D. (2002). Correlation and dependency in risk management:

properties and pitfalls. In Dempster, M., editor, *Risk Management: Value at Risk and Beyond*, pages 176–223. Cambridge University Press, Cambridge.

[Fang et al., 1990] Fang, K.-T., Kotz, S., and Ng, K.-W. (1990). *Symmetric Multivariate and Related Distributions*. Chapman & Hall, London.

[Prause, 1999] Prause, K. (1999). *The generalized hyperbolic model: estimation, financial derivatives and risk measures*. PhD thesis, Institut für Mathematische Statistik, Albert-Ludwigs-Universität Freiburg.