On uniqueness of symmetric union diagrams

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joint work with Carlo Collari

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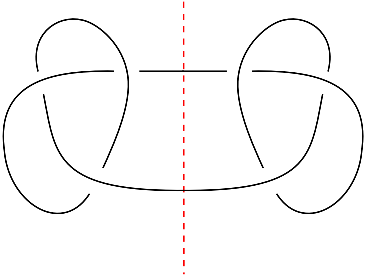
Background and motivation

Question 1 (Fox): $K \subset S^3$ slice knot $\Rightarrow$ K ribbon knot ?

Example of ribbon knot: $T \# T^*$



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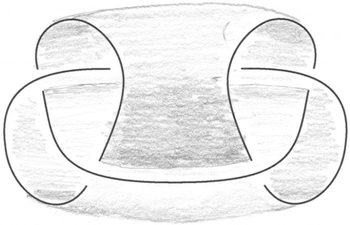

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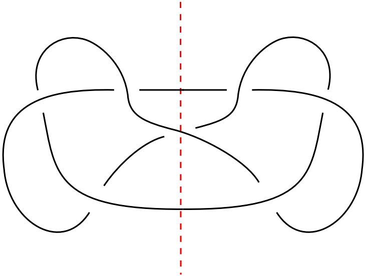
Another example of ribbon knot: $T \widetilde{\#} T^*$



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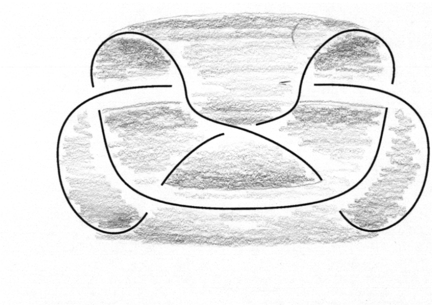
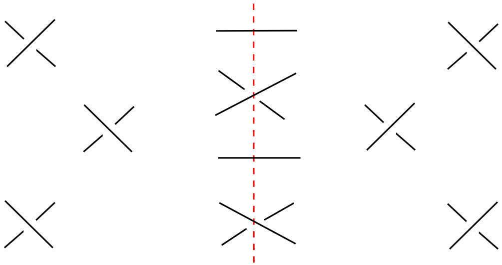


The trefoil $T = T(2,3)$ is the partial knot

Background and motivation

In general: a symmetric union diagram [1] yields a ribbon knot



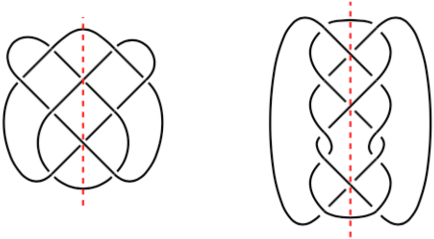
The partial knot is always defined

Question 2 (The existence problem): $K \subset S^3$ ribbon knot
$\Rightarrow$ does K have a symmetric union diagram ?

[1] Kinoshita and Terasaka (1957)

Question 3 (The uniqueness problem [2]):
If $K \subset S^3$ has a symmetric union diagram,
can K have two "distinct" symmetric union diagrams ?

Example: symmetric union diagrams of the knot 8_9



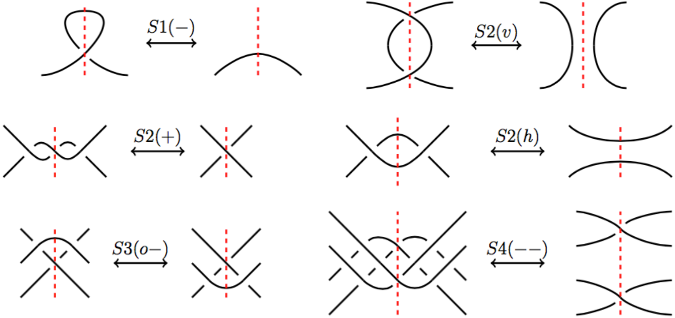
Partial knots: figure-8 (left) and $T(2,5)$ (right)

[2] Eisermann and Lamm (2007)

Background and motivation

Definition (Eisermann-Lamm): symmetric Reidemeister moves

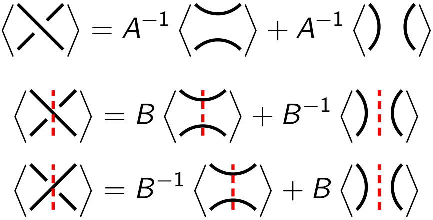
Reidemeister moves $R1$, $R2$, $R3$ performed symmetrically $+$



Note: each move preserves the partial knot

Background and motivation

refined Jones polynomial (Eisermann and Lamm):



$$\langle C \rangle = (-A^2 - A^{-2})^{n-m}(-B^2 - B^{-2})^{m-1}$$

C $=$ {n circles intersecting the axis in 2m points}

$$W_D(A,B) = (-A^{-3})^{\alpha(D)}(-B^{-3})^{\beta(D)}\langle D \rangle \in \mathbb{Z}(A,B)$$

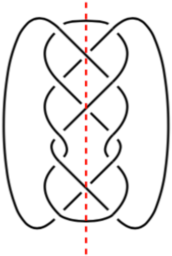
$\alpha(D) =$ off-axis writhe, $\beta(D) =$ on-axis writhe

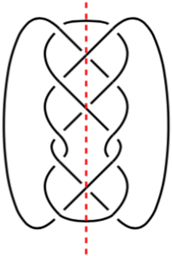
$W_D(t,t) = V(t) =$ Jones polynomial

Applications of the refined Jones polynomial (E.-L.):

Diagrams of 8_9 with the same partial knots and different W's:

D_{8_9} and D'_{8_9}:



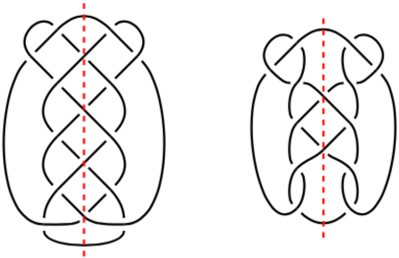


Background and motivation

Applications of the refined Jones polynomial (E.-L.):

$\exists$ 2-bridge knots diagrams D_n, D'_n, $n \geq 2$, with

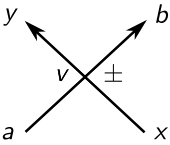
- same partial knots
- $D_2 = D_{8_9}$, $D'_2 = D_{8_9}$,
- $W_{D_n} \neq W_{D'_n}$ if $n \neq 4$

D_4 and D'_4: 

Knot invariants from lattice models (Jones): $Z = \sum W$

Vertex models

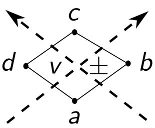
$$Z_{\text{vertex}} = \sum_{\{\text{edge states}\}} \prod_v \omega_v^{\pm} \left(\begin{smallmatrix} y & b \\ z & x \end{smallmatrix} \right)$$



"Enhanced" vertex models + conditions (EYBE) $\rightsquigarrow$ knot invariants (recover Homfly, Kauffman)

IRF models

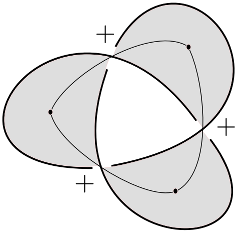
$$Z_{\text{IRF}} = \sum_{\{\text{face states}\}} \prod_v B_v^{\pm}(a, b, c, d)$$

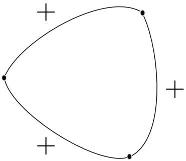


"Enhanced" IRF models $\leftrightarrow$ "special" E.V.M. $(a + x = y + b)$ $\rightsquigarrow$ knot invariants (recover Homfly, Kauffman)

Edge-interaction models

Edge-interaction models (a.k.a. spin models)
$\leftrightarrow$ "special" IRF models :

$D =$ 

$\Gamma_D =$ 

medial graph

$$X = \{1, \ldots, n\}, \; d \in \{\pm\sqrt{n}\}, \; N = |\Gamma_D^0|$$

$$Z_{\text{spin}} = d^{-N} \sum_{\sigma:\ \Gamma_D^0 \to X} \; \prod_{e \in \Gamma_D^1} W^{\pm}(e, \sigma)$$

Spin models

$$Z_{\mathrm{spin}} = d^{-N} \sum_{\sigma:\ \Gamma_D^0 \to X} \prod_{e \in \Gamma_D^1} W^{\pm}(e,\sigma)$$

$$W^{\pm}(e,\sigma) = W^{\pm}(\sigma(e_1),\sigma(e_2)),\ \partial e = \{e_1, e_1\}$$

$$W^{\pm} = \text{symmetric } n \times n \text{ matrix}$$

- $R2$-invariance of Z_{spin}: $W^+ \circ W^- = J$ (all-1 matrix)
- $R3$-invariance of Z_{spin}:

$$\begin{cases} W^+ Y_{a,b} = dW^-(a,b) Y_{a,b}, & Y_{a,b} \in \mathbb{C}^n \\ Y_{a,b}(x) = W^+(x,a)/W^+(x,b) \end{cases}$$

- $R1$-invariance: $I_D(W^+, d) = W^+(x,x)^{-w(D)} Z_{\mathrm{spin}}$

($W^+(x,x)$ independent of x)

Examples of spin models

- Potts model: $\xi \in \mathbb{C},\ \xi^8 + (2-n)\xi^4 + 1 = 0,\ n \geq 2$

$W^+_{\text{Potts}} := (-\xi^{-3})I + \xi(J - I),\ d := -\xi^2 - \xi^{-2} \in \{\pm\sqrt{n}\}$

- Pentagonal model:

$$W^+_{\text{pent}} = \begin{pmatrix} 1 & \omega & \omega^{-1} & \omega^{-1} & \omega \\ \omega & 1 & \omega & \omega^{-1} & \omega^{-1} \\ \omega^{-1} & \omega & 1 & \omega & \omega^{-1} \\ \omega^{-1} & \omega^{-1} & \omega & 1 & \omega \\ \omega & \omega^{-1} & \omega^{-1} & \omega & 1 \end{pmatrix}, \quad \omega = e^{2\pi i/5}, \quad d = \sqrt{5}$$

(De La Harpe):

$\frac{1}{d} I(W^+_{\text{Potts}}, d) = V(\xi^4)$ ($V =$ Jones)

$\frac{1}{d} I(-iW^+_{\text{pent}}, -d) = F(-i, 2i\cos(2\pi/5))$ ($F =$ Kauffman)

Refined spin models – role of the axis

$$Z_{\text{spin}} = d^{-N} \sum_{\sigma:\ \Gamma_D^0 \to X} \prod_{e \in \Gamma_D^1} W^{\pm}(e,\sigma)$$

$$\Gamma_D^1 \ \leftrightarrow \ \{\text{crossings}\}$$

Idea:

$$\Gamma_A^1 \cup \Gamma_B^1 \leftrightarrow \{\text{crossings on the axis}\} \cup \{\text{crossings off the axis}\}$$

$$\widetilde{Z}_{\text{spin}} = d^{-N} \sum_{\sigma:\ \Gamma_D^0 \to X} \prod_{e \in \Gamma_A^1} V^{\pm}(e,\sigma) \prod_{e \in \Gamma_B^1} W^{\pm}(e,\sigma)$$

$V^{\pm} =$ symmetric $n \times n$ matrix

Refined spin models – the Nomura algebra

Can we choose $V^{\pm}$ so that $\widetilde{Z}_{\text{spin}}$ is invariant under SR moves ?

Nomura algebra:

$$N_{W^+} = \{A \in M_n(\mathbb{C}) \mid Y_{a,b} \ A\text{-eigenvector } \forall a, b\} \subseteq M_n(\mathbb{C})$$

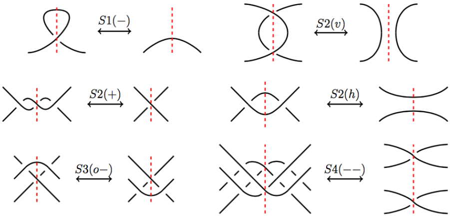
- closed under Hadamard product $\circ$ and transposition τ
- self-dual: $\psi\colon \ N_{W^+} \to M_n(\mathbb{C}), \ AY_{a,b} = \psi(A)(a,b)Y_{a,b}$

$\Rightarrow \ \psi|_{N_{W^+}}\colon \ N_{W^+} \overset{\cong}{\to} N_{W^+}, \ \psi^2|_{N_{W^+}} = n\tau|_{N_{W^+}}$

- $I, J, \pm W^{\pm} \in N_{W^+}$

Examples:
- $N_{W^+_{\text{Potts}}} = \langle I, J - I \rangle \subset M_n(\mathbb{C})$ ("smallest")
- $N_{W^+_{\text{pent}}} = \left\langle I, A_1 = \begin{pmatrix} 0&1&0&0&1\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&1&0&1\\ 1&0&0&1&0 \end{pmatrix}, A_2 = \begin{pmatrix} 0&0&1&1&0\\ 0&0&0&1&1\\ 1&0&0&0&1\\ 1&1&0&0&0\\ 0&1&1&0&0 \end{pmatrix} \right\rangle \subset M_5(\mathbb{C})$

Refined spin models – choice of $V^{\pm}$



Theorem (Collari-L.): Let $V^{\pm} \in N_{W^+}$. Then,

- S2($\pm$) and $S2(h)$-invariance of $\widetilde{Z}_{\text{spin}}$: automatic
- S2(v)-invariance of $\widetilde{Z}_{\text{spin}}$: $V^+ \circ V^- = J$
- S3 and S4-invariance of $\widetilde{Z}_{\text{spin}}$: $\psi(V^+) = dV^-$
- S1-invariance: $I_D(W^+, V^{\pm}, d) = V^+(x,x)^{-w_A(D)} \widetilde{Z}_{\text{spin}}$

Refined spin models – choice of $V^{\pm}$

Theorem (Collari-L.): Let $V^+ \in N_{W^+}$. Then,

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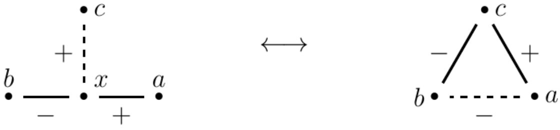
($V^+(x,x)$ independent of x)

Remark: $\pm W^{\pm}_{\text{Potts}} \in \langle I, J \rangle \subset N_{W^+} \Rightarrow$ system $\begin{cases} V^+ \circ V^- = J \\ \psi(V^+) = dV^- \end{cases}$
always has solutions, giving Potts-refined spin models

Sketch: $\psi(V^+) = dV^- \Rightarrow$ S3-invariance

Local change of medial graphs under an $S3$-move:



$$\Longleftrightarrow$$

$$\sum_{x \in X} V^+(x,c)W^+(x,a)W^-(b,x) = dV^-(a,b)W^+(c,a)W^-(b,c)$$

$$\Longleftrightarrow$$

$$\psi(V^+) = dV^-$$

Applications – 1

Example: $N_{W^+_{\text{pent}}} = \langle I, A_1, A_2 \rangle$,

$$W^+_{\text{pent}} = I + \omega A_1 + \omega^4 A_2\,,\ \omega = e^{\frac{2\pi i}{5}}\,,\ d = \sqrt{5}$$

$$V^{\pm}_{a,b,c} = a^{\pm 1} I + b^{\pm 1} A_1 + c^{\pm 1} A_2\,,\ \psi(V^+) = dV^- \ \Leftrightarrow$$

$$(*)\ \begin{cases} a(a+2b+2c) = d \\ b(a+2(\omega^2+\omega^3)b+2(\omega+\omega^4)c) = d \\ c(a+2(\omega+\omega^4)b+2(\omega^2+\omega^3)c) = d \end{cases}$$

$(a,b,c) \in \{\pm\frac{\sqrt{d}}{2}i(-2,1,1), \pm\sqrt{\frac{d}{3}}i(-1,1,1)\}$ satisfy $(*)$ and

$$I_{D_2}(W^+_{\text{pent}}, V^{\pm}_{a,b,c}, \sqrt{5}) \neq I_{D_2'}(W^+_{\text{pent}}, V^{\pm}_{a,b,c}, \sqrt{5})$$

$\Rightarrow\ D_2\,,\ D_2'$ not SR equivalent

Applications – 2

- $I_{D\#\widetilde{D}}(W^+_{\text{pent}}, V^\pm_{a,b,c}, \sqrt{5}) =$
$$\frac{1}{d} I_D(W^+_{\text{pent}}, V^\pm_{a,b,c}, \sqrt{5}) I_{\widetilde{D}}(W^+_{\text{pent}}, V^\pm_{a,b,c}, \sqrt{5})$$

$\Rightarrow$ $\#^k D_2$, $\#^k D_2'$ not SR equivalent

- Dropping condition $V^+ \circ V^- = J$ one looses S2(v)-invariance, but can find many $V^+, V^- \in N_{W^+_{\text{Potts}}}$ such that $\psi(V^+) = dV^-$,

$$I_{D_4}(W^+_{\text{Potts}}, V^\pm, d) \neq I_{D_4'}(W^+_{\text{Potts}}, V^\pm, d)$$

$\Rightarrow$ D_4, D_4' cannot be proved SR equivalent without S2(v)-moves

Applications – 3

- Refined cyclic models $\{W^+_{c,n}\}_{n\geq 3}$, $W^+_{c,n} \in M_n(\mathbb{C})$

$I^c_D(n) := I_D(W^+_{c,n}, V^{\pm}_{c,n}, d_n)$

Diagrams	Distinct $I^c_n(D), 1 \leq n \leq 10$
D_2, D'_2	$I^c_{D_2}(5) \neq I^c_{D'_2}(5)$
D_3, D'_3	$I^c_{D_3}(7) \neq I^c_{D'_3}(7)$
D_4, D'_4	–
D_5, D'_5	–
D_6, D'_6	–

Applications – 3

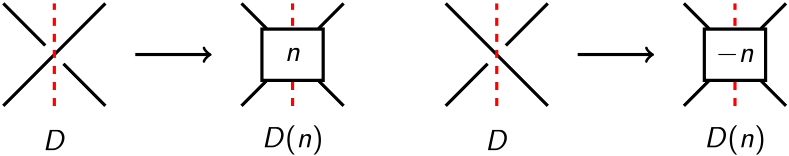
- Refined cyclic models $\{W^+_{c,n}\}_{n\geq 3}$, $W^+_{c,n} \in M_n(\mathbb{C})$

$I^c_D(n) := I_D(W^+_{c,n}, V^{\pm}_{c,n}, d_n)$

Diagrams	Distinct $I^c_D(n), 1 \leq n \leq 10$	$H_1(\Sigma_2(K))$
$D_2,\ D_2'$	$I^c_{D_2}(5) \neq I^c_{D_2'}(5)$	$\mathbb{Z}/25\mathbb{Z}$
$D_3,\ D_3'$	$I^c_{D_3}(7) \neq I^c_{D_3'}(7)$	$\mathbb{Z}/49\mathbb{Z}$
$D_4,\ D_4'$	–	$\mathbb{Z}/81\mathbb{Z}$
$D_5,\ D_5'$	–	$\mathbb{Z}/121\mathbb{Z}$
$D_6,\ D_6'$	–	$\mathbb{Z}/169\mathbb{Z}$

Question left open: are D_4, D_4' SR equivalent ?

A different approach to SR equivalence



Proposition (Collari-L.):

- $W_{D_4(n)} = W_{D_4'(n)}$
- $I_{D_4(n)}(W^+, W^{\pm}_{\text{Potts}}, d) = I_{D_4'(n)}(W^+, W^{\pm}_{\text{Potts}}, d)$

Theorem (Collari-L.):

- $D\,,\ D'$ SR equivalent $\Rightarrow$ $D(n)\,,\ D'(n)$ SR equivalent
- $D_4(2)\,,\ D_4'(2)$ are not Reidemeister equivalent

$\Rightarrow$ $D_4\,,\ D_4'$ are not SR equivalent

$D_4(2)$, $D_4'(2)$ are not Reidemeister equivalent

- K_1, K_1' have distinct third cyclic branched covers:

$$H_1(\Sigma_3(K_1);\mathbb{Z}) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$$

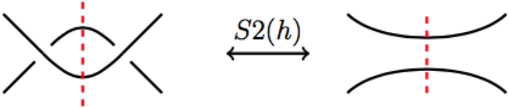
$$H_1(\Sigma_3(K_1');\mathbb{Z}) \cong \mathbb{Z}/49\mathbb{Z} \oplus \mathbb{Z}/49\mathbb{Z}.$$

$K_s = \mathrm{knot}(D_{4s}(2))$, $K_s' = \mathrm{knot}(D_{4s}'(2))$:

- K_s, K_s' have the same Alexander polynomials but distinct second Alexander ideals for each $s \geq 1$

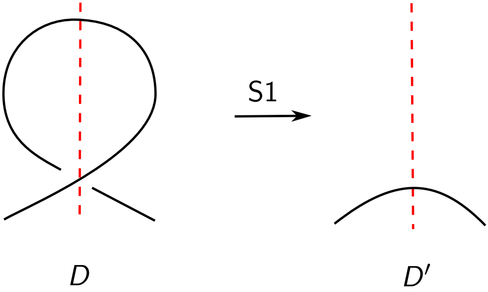
$\Rightarrow$ D_{4s}, D_{4s}' not SR equivalent $\forall s \geq 1$

D, D' S2(h) equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S2(h) equivalent

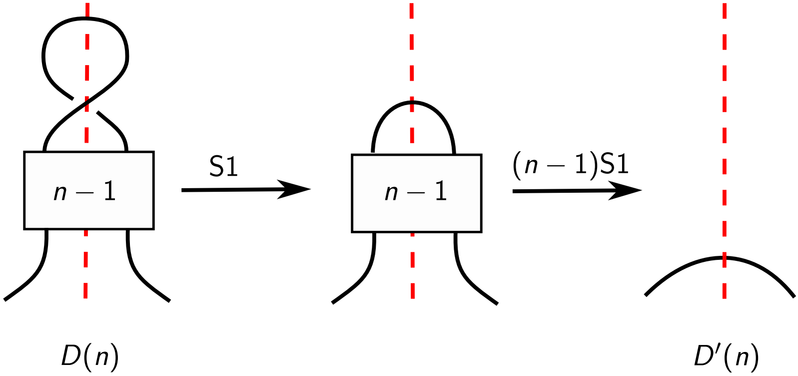


$D = D(n)$ $\qquad$ $D' = D'(n)$

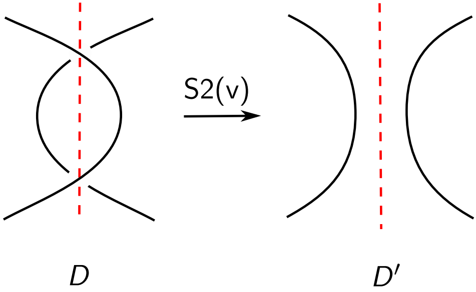
D, D' S1 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S1 equivalent



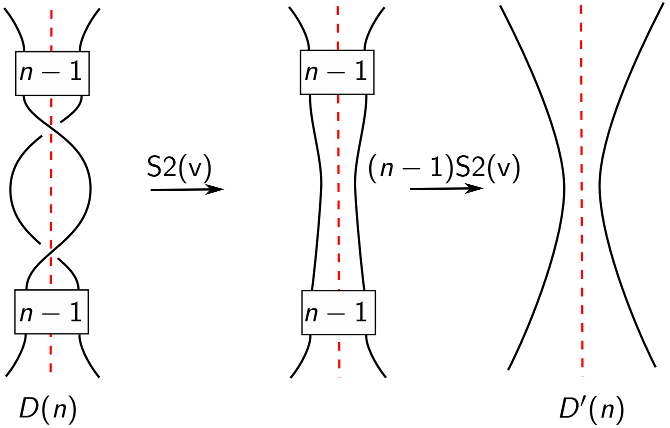
D, D' S1 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S1 equivalent



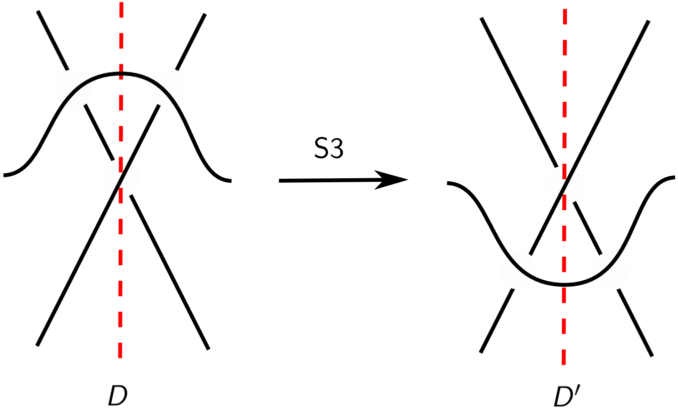
D, D' S2(v) equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S2(v) equivalent



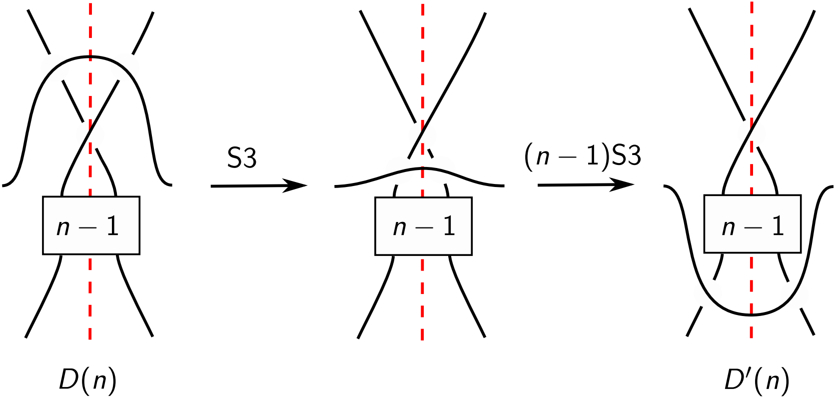
D, D' S2(v) equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S2(v) equivalent



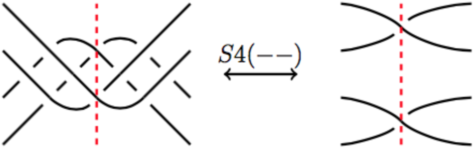
D, D' S3 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S3 equivalent

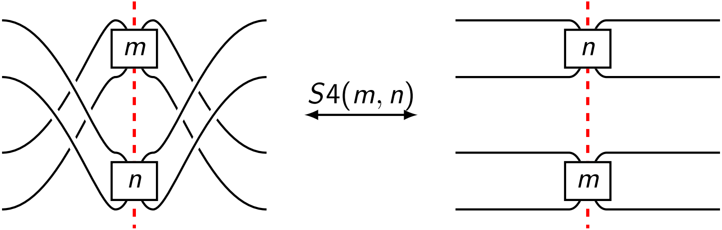


D, D' S3 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ S3 equivalent

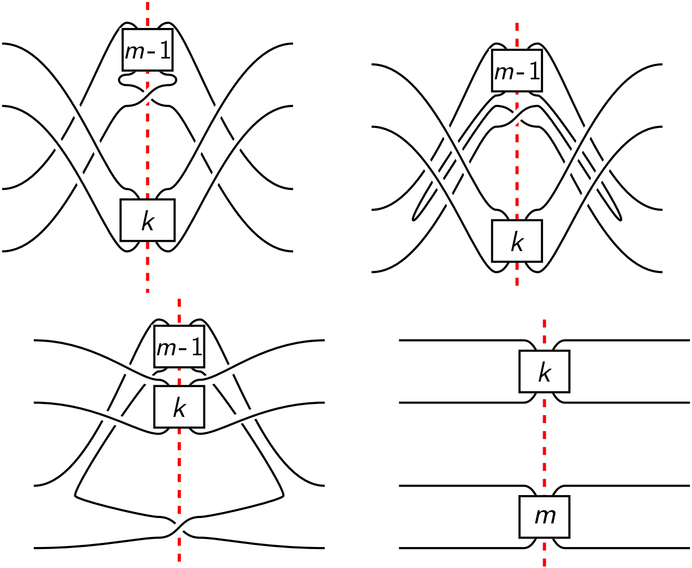


D, D' S4 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ SR equivalent

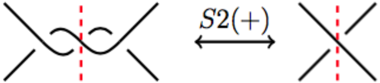


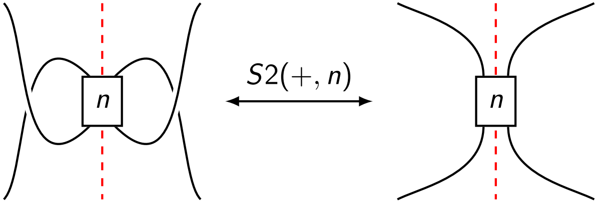


Double induction: case $S(m, k)$, $1 \leq k < n$

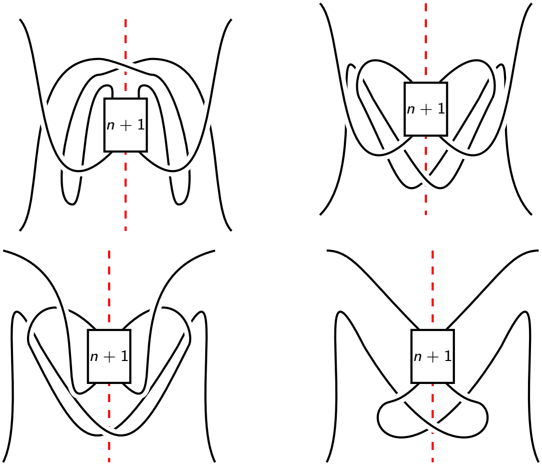


D, D' S2 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ SR equivalent





D, D' S2 equivalent $\Rightarrow$ $D(n)$, $D'(n)$ SR equivalent



Thank you for listening !