Knotted spheres in 4- and 5-manifolds

Swiss Knots, July 19, 2019

Peter Teichner, MPIM Bonn
joint work with Rob Schneiderman

Knot theory $\underset{①}{:=}$ study of the set

$$\{ S^1 \hookrightarrow \mathbb{R}^3 \} / \text{isotopy}$$

or more generally

$$\underset{②}{:=} \{ S^1 \hookrightarrow N^3 \} / \text{isotopy} \supseteq \left\{ \begin{matrix} S^1 \hookrightarrow N^3 \text{ in a} \\ \text{given homotopy class} \end{matrix} \right\} / \text{isotopy}$$

or more generally

huge because $2+2=4$

$$\underset{③}{:=} \left\{ \begin{matrix} S^m \hookrightarrow N^n \text{ in a} \\ \text{given homotopy class} \end{matrix} \right\} / \text{isotopy}$$

or more generally .....

Today: Complete **knot invariant** for $m=2$ and $n=4, 5$.

No knotted surfaces: F^2 closed connected,
[Whitney-Wu 1958] N^5 **simply-connected** (smooth)

For $K, K' : F^2 \hookrightarrow N$, **homotopy $\Rightarrow$ isotopy**.

Rem.: At Swiss **links**, I would have added linking numbers as obstructions to isotopy.

Outline of proof: One turns a generic track

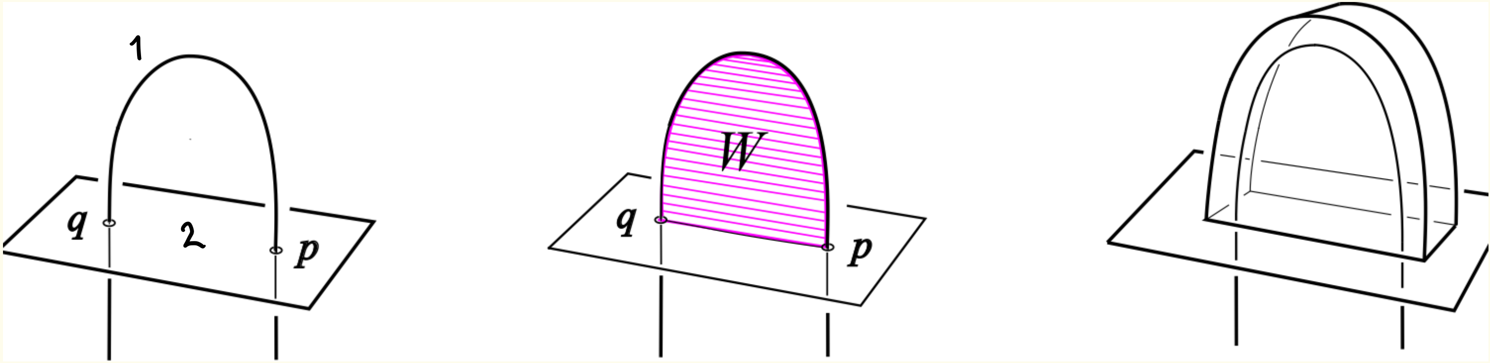
only transverse double points
$3+3=6$

$H : F^2 \times [0,1] \looparrowright N^5 \times [0,1]$

into an embedding (rel. ∂)

by Whitney moves + cusp moves, preserving level sets. $\blacksquare$

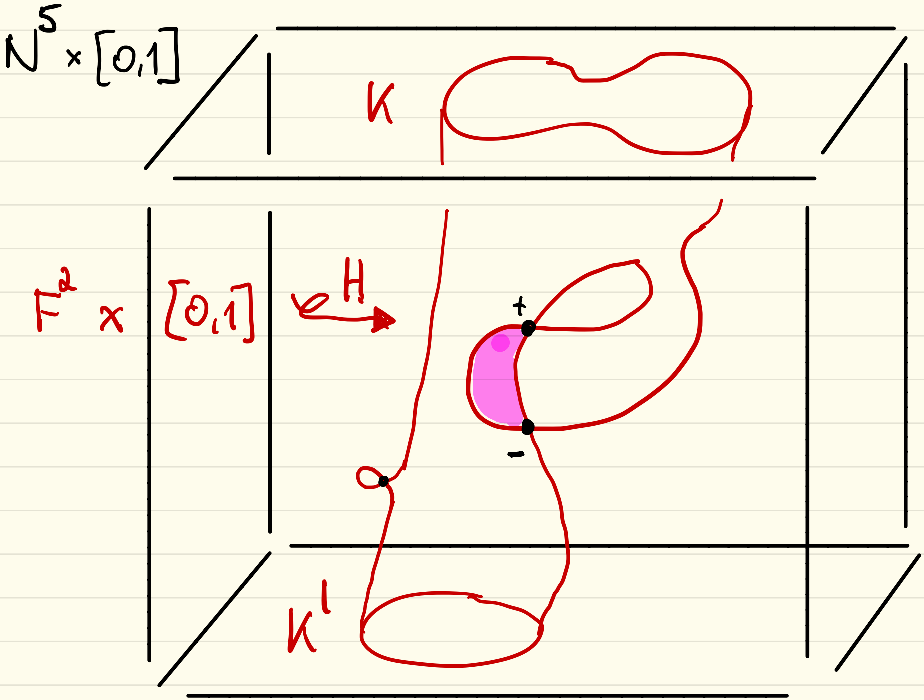
Whitney move in $\mathbb{R}^3$ is pictured below:
First sheet is 1-dim., second sheet is 2-dim,
a pair of double points p, q are cancelled.



This movie also gives the 6-dim. case by crossing the ambient space with $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ and the sheets with $0 \times \mathbb{R}^2$ resp. $\mathbb{R} \times 0$.

Schematic picture of a generic

track $H: F^2 \times [0,1] \hookrightarrow N^5 \times [0,1]$



Proof of W-W:

- $3 + 3 = 6 \Longrightarrow$

H has only double points

- add cusps until signed sum is zero
- do Whitney moves to cancel all double points: $\pi_1 N = \{1\}$ and

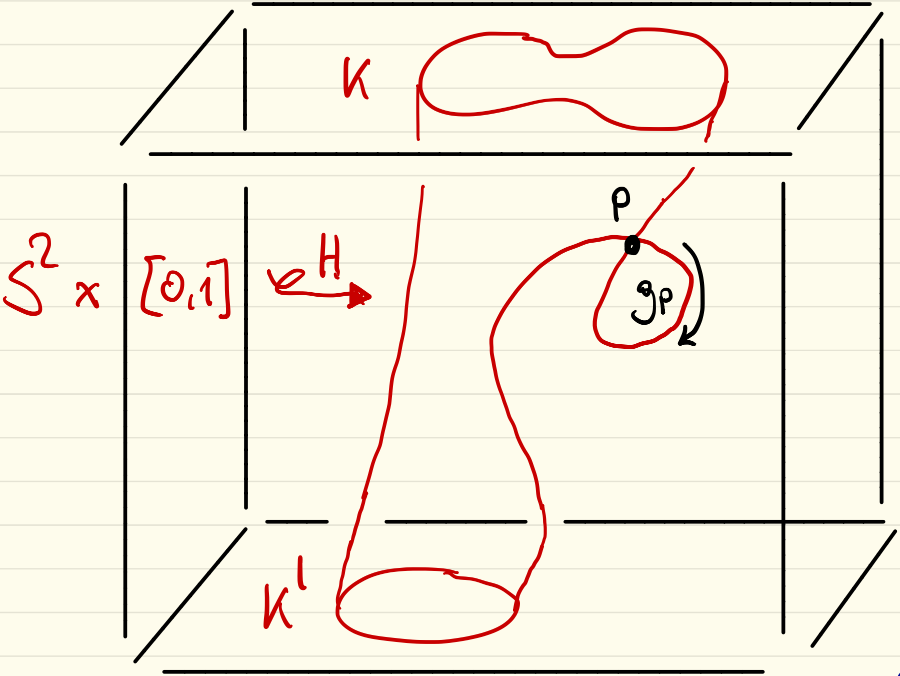
$$2 + 2 < 6,$$

$$3 + 2 < 6 \quad !$$

Self-intersection invariant for

3-manifolds in 6-manifolds: $F^2 = S^2$

$\pi_1 N^5 \neq \{1\}$



$\Rightarrow \mu(H) := \sum_{\substack{\text{double}\\\text{points}\\p \text{ of } H}} \varepsilon_p \cdot g_p$ is well-defined

in $\dfrac{\mathbb{Z}[\pi_1 N]}{\langle g + g^{-1}, 1 \rangle}$

$\varepsilon_p \in \{\pm 1\}$, $g_p \in \pi_1 N$.

1-connected proof works

$\Leftrightarrow \mu(H) = 0$.

$$\pi_2^{emb} N := \left\{ \frac{\text{based embeddings}}{\text{based isotopy}} \ S^2 \hookrightarrow N^5 \right\} \xrightarrow{p} \pi_2 N$$

$\pi_1 N$-module

$$\overset{\pi_2^{emb} N}{\underset{F}{\Psi}}$$

Fix F

and study the fibers $\mathcal{R}_F := p^{-1}(p(F))$ $\ni (K, H)$

$$\downarrow \text{I}$$

$$[\mu(H)]$$

Theorem 1: [S-T, 2019]

$$\mathcal{R}_F \overset{\cong}{\downarrow} W$$

$$\mathcal{A}_F := \frac{\mathbb{Z}[\pi_1 N] / \langle g + g^{-1}, 1 \rangle}{(\mu + \lambda_N(F, -))(\pi_3 N)}$$

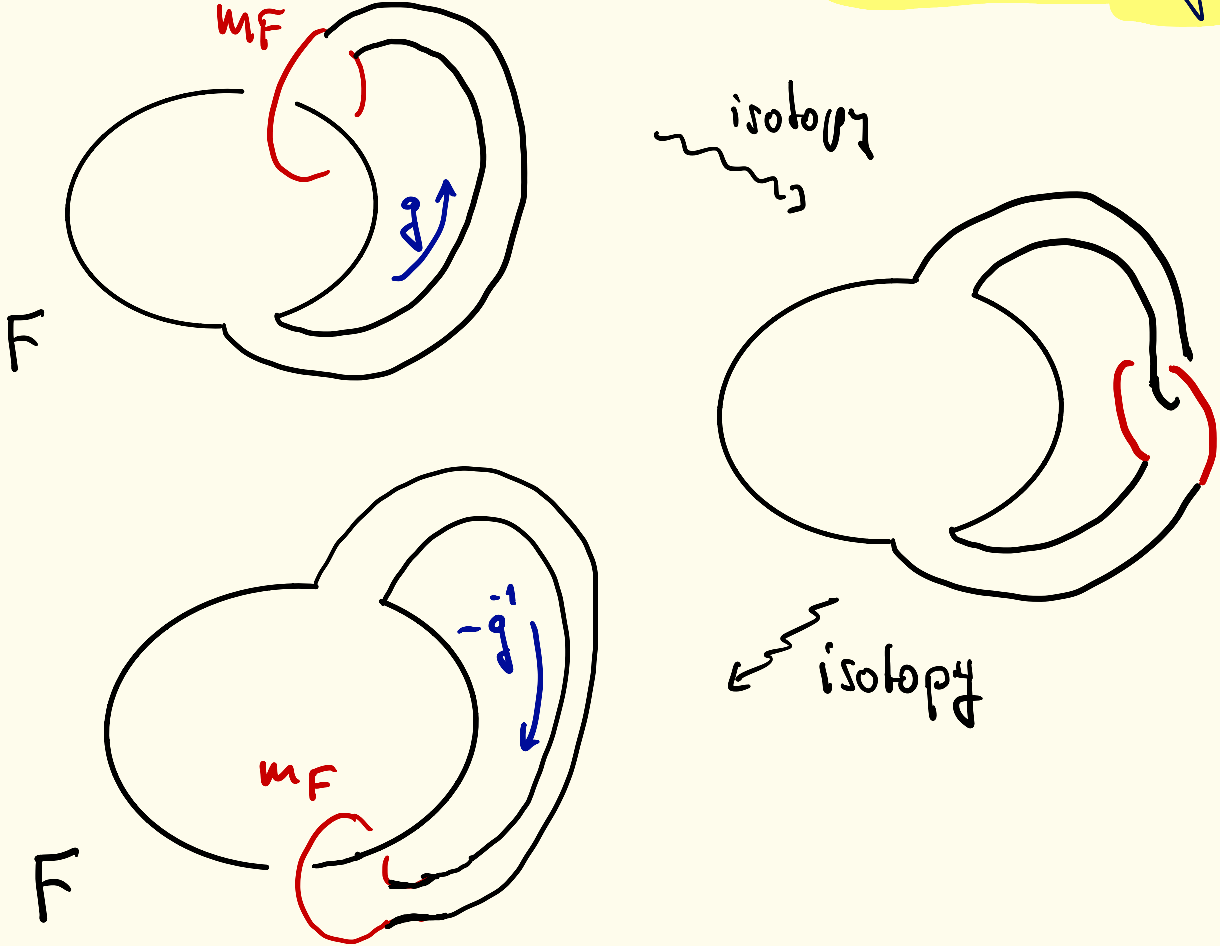
Idea for proof:

W injective ✓ surjective:

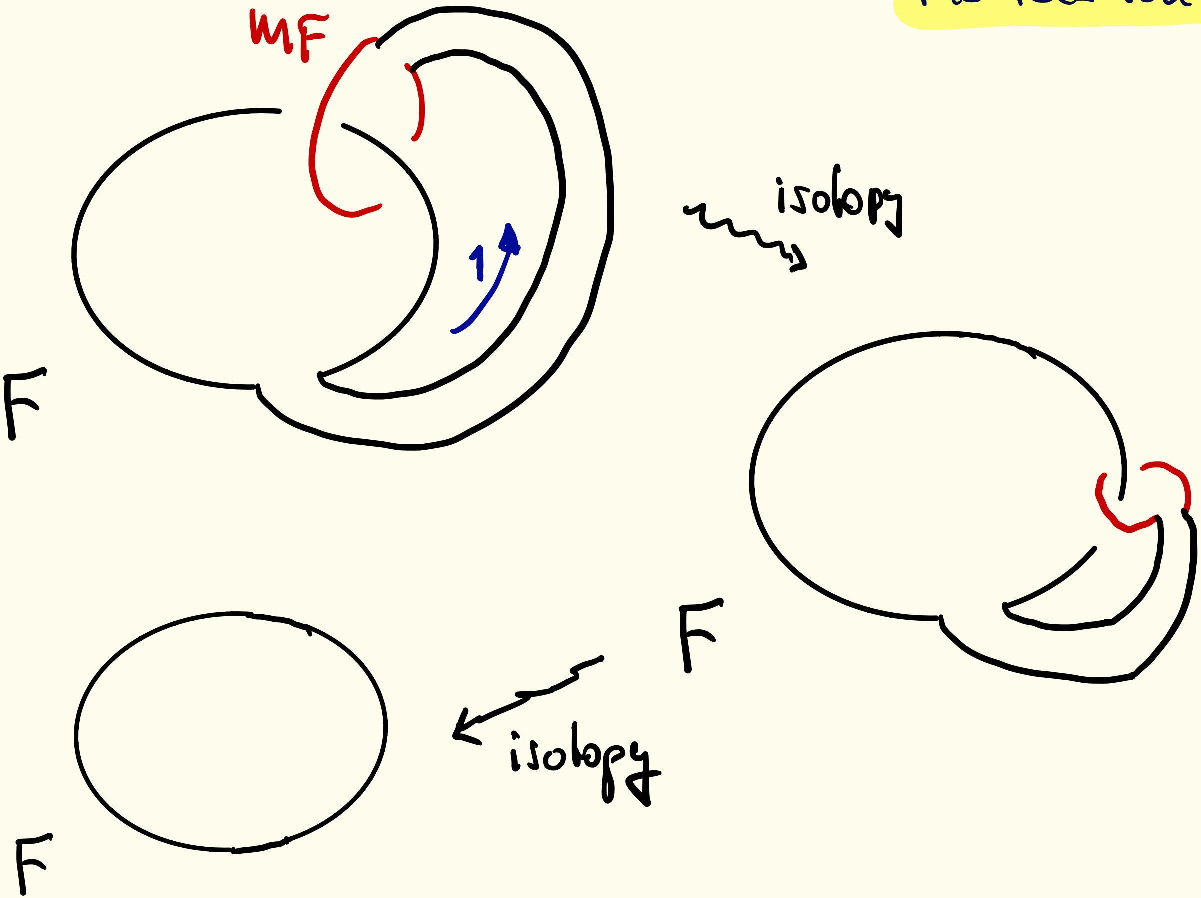
$\exists$ geometric action of $\mathcal{A}_F$ on $\mathcal{R}_F$ by adding meridian spheres to F ∎

independence of choice of homotopy H: "Add" generic 3-spheres to H.

The relation $g + g^{-1} = 0$.



The relation $1 = 0$.



Reidemeister Theorem

In $N^3 = M^2 \times \mathbb{R}$, knot theory translates to:

(i) Any $F: S^1 \hookrightarrow M^2 \times \mathbb{R}$ is isotopic to (f, b) where

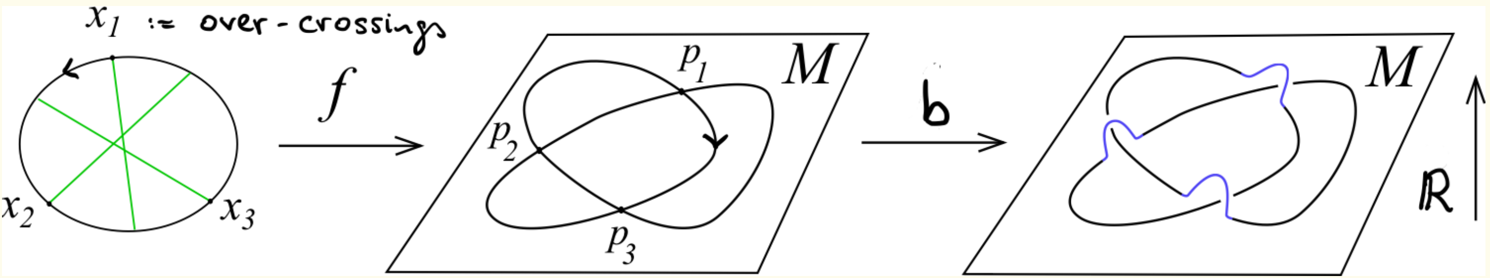
- $f: S^1 \hookrightarrow M^2$ has only transverse double points $\{p_1, \dots, p_n\} \subseteq M$,
- $b: S^1 \to \mathbb{R}$ is a bump fct. giving over/under data at p_i.

The isotopy class of (f, b) only depends on signs of $b(f^{-1}(p_i))$.

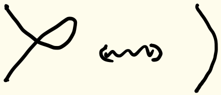
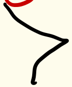
(ii) (f, b) is isotopic to (f', b') if and only if f & f' are related by a finite sequence of isotopies and type I + II + III moves taking over/under data from b to b'.

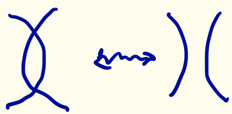
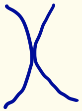
Reidemeister

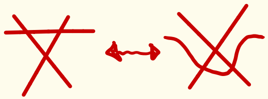
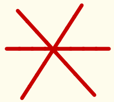
detects homotopy versus isotopy!



(iii) (f,b) is homotopic to (f',b') if and only if f & f' are related by a finite sequence of isotopies and type I + II + III moves: These correspond to the singularities of a generic track $S^1 \times [0,1] \hookrightarrow M^2 \times [0,1]$:

 I : cusp , i.e. non-immersion point

 II : tangency , will turn to Whitney move

 III : triple point 

5-dim. Reidemeister Theorem

Corollary: For $N^5 = M^4 \times \mathbb{R}$ have

(i) Any $F: S^2 \hookrightarrow M^4 \times \mathbb{R}$ is isotopic to (f, b) where

- $f: S^2 \looparrowright M^4$ has only transverse double points $\{p_1, \dots, p_n\} \subseteq M$,
- $b: S^2 \to \mathbb{R}$ is a bump fct. giving over/under data at p_i.

The isotopy class of (f, b) only depends on signs of $b(f^{-1}(p_i))$.

(ii) (f, b) is isotopic to (f', b') if and only if f & f' are related by a finite sequence of isotopies and type I + II moves taking over/under data from b to b'.

cusps + Whitney move

$(\Leftrightarrow \; f \simeq f')$

No triple points in a track

$S^2 \times [0,1] \hookrightarrow M^4 \times [0,1]$!

Remark: Let's use the 4-dim. result that

$f \simeq f' : S^2 \hookrightarrow M^4 \iff f$ & f' are related by a finite sequence of isotopies, cusps & Whitney moves $\Rightarrow$

$F \simeq F' : S^2 \hookrightarrow M^4 \times \mathbb{R} \iff \exists f$ s.t. F isotopic to (f, b)

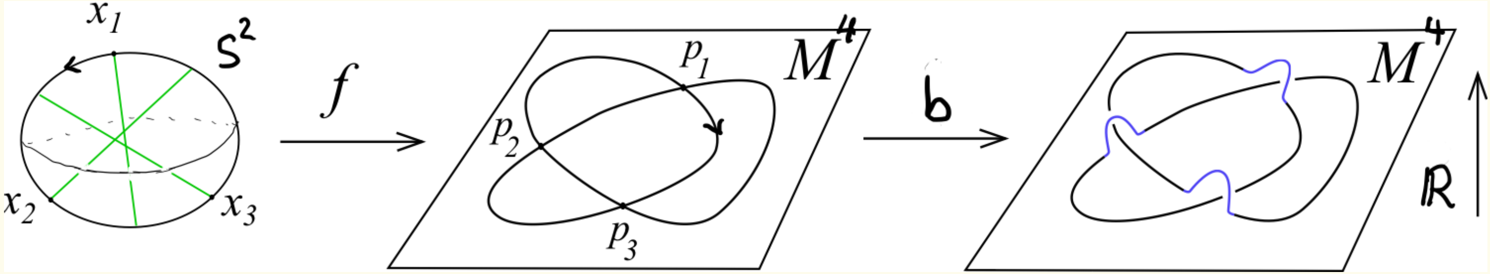
and F' — " — (f, b').

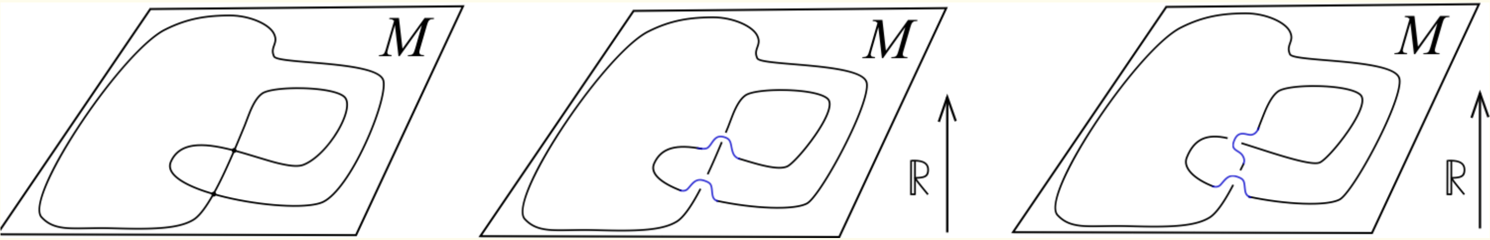
Lemma: The self-intersections of f classify, i.e. the Whitney-Wu invariant is given by

$$W((f,b),(f,b')) = \sum_{\substack{p_i \text{ s.t.} \\ x_i \neq x_i'}} \varepsilon_i(f) \cdot g_{x_i}(f) \in \mathcal{A}_{(f,b)}$$

$x_i \in f^{-1}(p_i)$ "over" sheet

and completely determines our isotopy classes ∎


x_1
S^2
x_2
x_3
f
p_1
p_2
p_3
M^4
b
M^4
$\mathbb{R}$


M
M
$\mathbb{R}$
M
$\mathbb{R}$

Embedded spheres in 4-manifolds

Assume that f & $f' : S^2 \hookrightarrow M^4$ are homotopic via $H : S^2 \times [0,1] \hookrightarrow M^4 \times [0,1] \times \mathbb{R}$ $\overset{\wedge\!|}{M} \times \mathbb{R}$, i.e. $b = b' = 0$.

Lemma: $\mu(H) - \overline{\mu(H)} = \lambda(H,H) = 0 \in \mathbb{Z}[\pi_1 M]$

since we may push H into $\mathbb{R}$-direction. These are (self-) intersection of $H(S^2 \times [0,1])$ in $M^4 \times [0,1] \times \mathbb{R}$.

Corollary: $\mu(H) \in \mathbb{F}_2[I_M] \subseteq \dfrac{\mathbb{Z}[\pi_1 M]}{\langle g + g^{-1}, 1 \rangle}$

where $I_M := \{ g \in \pi_1 M \setminus 1 \mid g^2 = 1 \}$ are involutions in $\pi_1 M$

Similarly, $\lambda(H, f) = 0$, so the Whitney-Wu inv. takes values in $\mathbb{F}_2[I_M] / \mu(\pi_3 M)$.

Theorem 2: (Generalization of David Gabai's light-bulb trick in dim. 4)
[S-T, 2018]

Assume that $f: S^2 \hookrightarrow M^4$ has a **framed dual** $g: S^2 \hookrightarrow M$ in the sense that $f \pitchfork g = \{pt\}$.

$$\mathcal{R}^g_f(M) := \{ K : S^2 \hookrightarrow M^4 \mid K \simeq f \,,\, K \pitchfork g = \{pt\} \} / \text{isotopy}$$

Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{R}^g_f(M) & \rightarrowtail & \mathcal{R}_{f,0}(M \times \mathbb{R}) \\ W \downarrow \cong & & W \downarrow \cong \\ \dfrac{\mathbb{F}_2[T_M]}{\mu(\pi_3 M)} & \subseteq & \dfrac{\mathbb{Z}[\pi_1 M] / \langle g + g^{-1}, 1 \rangle}{\mu(\pi_3 M)} \end{array}$$

Has **huge cokernel**: free abelian group on the set $\dfrac{\{ g \in \pi_1 M \mid g^2 \neq 1 \}}{g \sim g^{-1}}$