Background material

1.1 Graph theory

1.1.1 Basics

For a simple graph $G = (V, E)$, we denote by $\overline{G} = (V, \overline{E})$ the complementary graph.

For $X \subseteq V$, we denote by $N(X)$ the set of neighbors of X, i.e., the set of nodes $j \in V$ for which there is a node $i \in V$ such that $ij \in E$. Warning: $N(X)$ may intersect X (if X is not a stable set).

For $X, Y \subseteq V$, we denote by $\kappa(X, Y)$ the maximum number of vertex disjoint paths from X to Y (if X and Y are not disjoint, some of these paths may be single nodes). By Menger's Theorem, $\kappa(X, Y)$ is the minimum number of nodes that cover all $X - Y$ paths in G. The graph G is k-connected if and only if $|V| > k$ and $\kappa(X, Y) = k$ for any two k-subsets X and Y. The largest k for which this holds is the *vertex-connectivity* of G, denoted $\kappa(G)$. The complete graph K_n is $(n-1)$-connected but not n-connected.

$B_G(v, t)$, where $v \in V$, denotes the set of nodes at distance at most t from v.

For $S \subseteq V$, we denote by $G[S] = (V, E[S])$ the subgraph induced by S.

1.2 Planar graphs

A graph $G = (V, E)$ is *planar*, if it can be drawn in the plane so that its edges are Jordan curves and they intersect only at their endnodes[1]. A *plane map* is a planar graph with a fixed embedding. We also use this phrase to denote the image of this embedding, i.e., the subset of the plane which is the union of the set of points representing the nodes and the Jordan curves representing the edges.

The complement of a plane map decomposes into a finite number of arcwise connected pieces, which we call the *faces* (or *countries*) of the planar map. We usually denote the number of nodes and edges of a graph G by n and m, respectively; if G is a plane map, then the number of its faces will be denotes by f.

[1] We use the word *node* for the node of a graph, the word *vertex* for the vertex of a polytope, and the word *point* for points in the plane or in other spaces.

Every planar map $G = (V, E)$ has a *dual map* $G^* = (V^*, E^*)$. As an abstract graph, this can be defined as the graph whose nodes are the faces of G. If the two faces share k edges, then we connect them in G^* by k edges, so that each edge $e \in E$ will correspond to an edge e^* of G^*. So $|E^*| = |E|$.

This dual has a natural drawing in the plane: in the interior of each face F of G we select a point v_F (which can be called its *capital* if we use the country terminology), and on each edge $e \in E$ we select a point u_e (this will not be a node of G^*, just an auxiliary point). We connect v_F to the points u_e for each edge on the boundary of F by nonintersecting Jordan curves inside F. If the boundary of F goes through e twice (i.e., both sides of e belong to F), then we connect v_F to u_e by two curves, entering e from two sides. The two curves entering u_e form a single Jordan curve representing the edge e^*. It is not hard to see that each face of G^* will contain a unique node of G, and so $(G^*)^* = G$.

A planar map is called a *triangulation* if every face has 3 edges. Note that a triangulation may have parallel edges, but no two parallel edges can bound a face. In every simple planar map we can introduce new edges to turn all faces into triangles while keeping the graph simple.

We often need the following basic fact about planar graphs:

Theorem 1.2.1 (Euler's Formula) *For every connected planar map, $n - m + f = 2$ holds.* □

Some important consequences of Euler's Formula are the following.

Corollary 1.2.2 (a) *A simple planar graph with n nodes has at most $3n - 6$ edges.*

(b) *A simple bipartite planar graph with n nodes has at most $2n - 4$ edges.*

(c) *Every simple planar graph has a node with degree at most 5.*

(d) *Every simple bipartite planar graph has a node with degree at most 3.* □

From (a) and (b) it follows immediately that the "Kuratowski graphs" K_5 and $K_{3,3}$ are not planar. This observation leads to the following characterization of planar graphs.

Theorem 1.2.3 (Kuratowski's Theorem) *A graph G is embedable in the plane if and only if it does not contain a subgraph homeomorphic to the complete graph K_5 or the complete bipartite graph $K_{3,3}$.* □

Among planar graphs, 3-connected planar graphs are especially important. We start with a simple but useful fact. A cycle C in a graph G is called *separating*, if $G \setminus V(C)$ has at least two connected components, where any chord of C is counted as a connected component here.

Proposition 1.2.4 *Let G be a 3-connected planar graph, and C a cycle in G. Then bounds a face if and only if it is non-separating.* □

Corollary 1.2.5 *Every simple 3-connected planar graph has an essentially unique embedding in the plane in the sense that the set of cycles that bound faces is uniquely determined.* □

1.3 Linear algebra

1.3.1 Basic facts about eigenvalues

Let A be an $n \times n$ real matrix. An *eigenvector* of A is a vector such that Ax is parallel to x; in other words, $Ax = \lambda x$ for some real or complex number λ. This number λ is called the *eigenvalue* of A belonging to eigenvector v. Clearly λ is an eigenvalue iff the matrix $A - \lambda I$ is singular, equivalently, iff $\det(A - \lambda I) = 0$. This is an algebraic equation of degree n for λ, and hence has n roots (with multiplicity).

The *trace* of the square matrix $A = (A_{ij})$ is defined as

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii}.$$

The trace of A is the sum of the eigenvalues of A, each taken with the same multiplicity as it occurs among the roots of the equation $\det(A - \lambda I) = 0$.

If the matrix A is symmetric, then its eigenvalues and eigenvectors are particularly well behaved. All the eigenvalues are real. Furthermore, there is an orthogonal basis $v_1, \ldots, v_n$ of the space consisting of eigenvectors of A, so that the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ are precisely the roots of $\det(A - \lambda I) = 0$. We may assume that $|v_1| = \cdots = |v_n| = 1$; then A can be written as

$$A = \sum_{i+1}^{n} \lambda_i v_i v_i^{\mathsf{T}}.$$

Another way of saying this is that every symmetric matrix can be written as $U^{\mathsf{T}} D U$, where U is an orthogonal matrix and D is a diagonal matrix. The eigenvalues of A are just the diagonal entries of D.

To state a further important property of eigenvalues of symmetric matrices, we need the following definition. A *symmetric minor* of A is a submatrix B obtained by deleting some rows and the corresponding columns.

Theorem 1.3.1 (Interlacing eigenvalues) *Let A be an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let B be an $(n-k) \times (n-k)$ symmetric minor of A with eigenvalues $\mu_1 \geq \cdots \geq \mu_{n-k}$. Then*

$$\lambda_i \leq \mu_i \leq \lambda_{i+k}.$$

We conclude this little overview with a further basic fact about nonnegative matrices.

Theorem 1.3.2 (Perron-Frobenius) *If an $n \times n$ matrix has nonnegative entries then it has a nonnegative real eigenvalue λ which has maximum absolute value among all eigenvalues. This eigenvalue λ has a nonnegative real eigenvector. If, in addition, the matrix has no block-triangular decomposition (i.e., it does not contain a $k \times (n-k)$ block of 0-s disjoint from the diagonal), then λ has multiplicity 1 and the corresponding eigenvector is positive.*

1.3.2 Semidefinite matrices

A symmetric $n \times n$ matrix A is called *positive semidefinite*, if all of its eigenvalues are nonnegative. This property is denoted by $A \succeq 0$. The matrix is *positive definite*, if all of its eigenvalues are positive.

There are many equivalent ways of defining positive semidefinite matrices, some of which are summarized in the Proposition below.

Proposition 1.3.3 *For a real symmetric $n \times n$ matrix A, the following are equivalent:*

(i) *A is positive semidefinite;*

(ii) *the quadratic form $x^T Ax$ is nonnegative for every $x \in \mathbb{R}^n$;*

(iii) *A can be written as the Gram matrix of n vectors $u_1, ..., u_n \in \mathbb{R}^m$ for some m; this means that $a_{ij} = u_i^{\mathsf{T}} u_j$. Equivalently, $A = U^{\mathsf{T}} U$ for some matrix U;*

(iv) *A is a nonnegative linear combination of matrices of the type xx^{T};*

(v) *The determinant of every symmetric minor of A is nonnegative.*

Let me add some comments. The least m for which a representation as in (iii) is possible is equal to the rank of A. It follows e.g. from (ii) that the diagonal entries of any positive semidefinite matrix are nonnegative, and it is not hard to work out the case of equality: all entries in a row or column with a 0 diagonal entry are 0 as well. In particular, the trace of a positive semidefinite matrix A is nonnegative, and $\mathrm{tr}(A) = 0$ if and only if $A = 0$.

The sum of two positive semidefinite matrices is again positive semidefinite (this follows e.g. from (ii) again). The simplest positive semidefinite matrices are of the form aa^{T} for some vector a (by (ii): we have $x^{\mathsf{T}}(aa^{\mathsf{T}})x = (a^{\mathsf{T}}x)^2 \geq 0$ for every vector x). These matrices are precisely the positive semidefinite matrices of rank 1. Property (iv) above shows that every positive semidefinite matrix can be written as the sum of rank-1 positive semidefinite matrices.

The product of two positive semidefinite matrices A and B is not even symmetric in general (and so it is not positive semidefinite); but the following can still be claimed about the product:

Proposition 1.3.4 *If A and B are positive semidefinite matrices, then $\mathrm{tr}(AB) \geq 0$, and equality holds iff $AB = 0$.*

Property (v) provides a way to check whether a given matrix is positive semidefinite. This works well for small matrices, but it becomes inefficient very soon, since there are many symmetric minors to check. An efficient method to test if a symmetric matrix A is positive semidefinite is the following algorithm. Carry out Gaussian elimination on A, pivoting always on diagonal entries. If you ever find a negative diagonal entry, or a zero diagonal entry whose row contains a non-zero, stop: the matrix is not positive semidefinite. If you obtain an all-zero matrix (or eliminate the whole matrix), stop: the matrix is positive semidefinite.

If this simple algorithm finds that A is not positive semidefinite, it also provides a certificate in the form of a vector v with $v^\mathsf{T} A v < 0$ (Exercise 1.3.6).

We can think of $n \times n$ matrices as vectors with n^2 coordinates. In this space, the usual inner product is written as $A \cdot B$. This should not be confused with the matrix product AB. However, we can express the inner product of two $n \times n$ matrices A and B as follows:

$$A \cdot B = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} B_{ij} = \mathrm{tr}(A^\mathsf{T} B).$$

Positive semidefinite matrices have some important properties in terms of the geometry of this space. To state these, we need two definitions. A *convex cone* in $\mathbb{R}^n$ is a set of vectors is closed under sum and multiplication positive scalars. Note that according to this definition, the set $\mathbb{R}^n$ is a convex cone. We call the cone *pointed*, if the origin is a vertex of it; equivalently, if it does not contain a line. Any system of homogeneous linear inequalities

$$a_1^\mathsf{T} x \geq 0, \ \ldots, \ a_m^\mathsf{T} x \geq 0$$

defines a convex cone; convex cones defined by such (finite) systems are called *polyhedral*.

For every convex cone C, we can form its *polar cone* C^*, defined by

$$C^* = \{x \in \mathbb{R}^n : x^\mathsf{T} y \geq 0 \ \forall y \in C\}.$$

This is again a convex cone. If C is closed (in the topological sense), then we have $(C^*)^* = C$.

The fact that the sum of two positive semidefinite matrices is again positive semidefinite (together with the trivial fact that every positive scalar multiple of a positive semidefinite matrix is positive semidefinite), translates into the geometric statement that *the set of all positive semidefinite matrices forms a convex closed cone $\mathcal{P}_n$ in $\mathbb{R}^{n \times n}$ with vertex* 0. This cone $\mathcal{P}_n$ is important, but its structure is quite non-trivial. In particular, it is non-polyhedral for $n \geq 2$; for $n = 2$ it is a nice rotational cone. For $n \geq 3$ the situation becomes more complicated, because $\mathcal{P}_n$ is neither polyhedral nor smooth: any matrix of rank less than $n-1$ is on the boundary, but the boundary is not differentiable at that point.

The polar cone of $\mathcal{P}$ is itself; in other words,

Proposition 1.3.5 *A matrix A is positive semidefinite iff $A \cdot B \geq 0$ for every positive semidefinite matrix B.*

Exercise 1.3.6 Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and let us carry out Gaussian elimination so that we always pivot on a nonzero diagonal entry as long as this is possible. Suppose that you find a negative diagonal entry, or a zero diagonal entry whose row contains a non-zero, stop: the matrix is not positive semidefinite. Construct a vector $v \in \mathbb{R}^n$ such that $v^{\mathsf{T}}Av < 0$.

1.3.3 Cross product

This construction is probably familiar from physics. For $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$, we define their *cross product* as the vector

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\cdot|\mathbf{b}|\cdot\sin\phi\cdot\mathbf{u}, \tag{1.1}$$

where ϕ is the angle between $\mathbf{a}$ and $\mathbf{b}$ ($0 \le \phi \le \pi$), and $\mathbf{u}$ is a unit vector in $\mathbb{R}^3$ orthogonal to the plane of $\mathbf{a}$ and $\mathbf{b}$, so that the triple $(\mathbf{a}, \mathbf{b}, \mathbf{u})$ is right-handed (positively oriented). The definition of $\mathbf{u}$ is ambiguous if $\mathbf{a}$ and $\mathbf{b}$ are parallel, but then $\sin\phi = 0$, so the cross product is 0 anyway. The length of the cross product gives the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

The cross product is distributive with respect to linear combination of vectors, it is anticommutative: $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$, and $\mathbf{a}\times\mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel. The cross product is not associative; instead, it satisfies the *Expansion Identity*

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}, \tag{1.2}$$

which implies the *Jacobi Identity*

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} + (\mathbf{b}\times\mathbf{c})\times\mathbf{a} + (\mathbf{c}\times\mathbf{a})\times\mathbf{b} = 0. \tag{1.3}$$

Another useful replacement for the associativity is the following.

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \det(\mathbf{a}, \mathbf{b}, \mathbf{c}) \tag{1.4}$$

(here $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the 3×3 matrix with columns $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

We often use the cross product in the special case when the vectors lie in a fixed plane Π. Let $\mathbf{k}$ be a unit vector normal to Π, then $\mathbf{a}\times\mathbf{b}$ is $A\mathbf{k}$, where A is the signed area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ (this means that T is positive iff a positive rotation takes the direction of $\mathbf{a}$ to the direction of $\mathbf{b}$, when viewed from the direction of $\mathbf{k}$). Thus in this case all the information about $\mathbf{a}\times\mathbf{b}$ is contained in this scalar A, which in tensor algebra would be denoted by $\mathbf{a}\wedge\mathbf{b}$. But not to complicate notation, we'll use the cross product in this case as well.

1.3.4 Matrices associated with graphs

Let G be a (finite, undirected, simple) graph with node set $V(G) = \{1, \dots, n\}$. The *adjacency matrix* of G is be defined as the $n \times n$ matrix $A_G = (A_{ij})$ in which

$$A_{ij} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

We can extend this definition to the case when G has multiple edges: we just let A_{ij} be the number of edges connecting i and j. We can also have weights on the edges, in which case we let A_{ij} be the weight of the edges. We could also allow loops and include this information in the diagonal, but we don't need this in this course.

The *Laplacian* of the graph is defined as the $n \times n$ matrix $L_G = (L_{ij})$ in which

$$L_{ij} = \begin{cases} d_i, & \text{if } i = j, \\ -A_{ij}, & \text{if } i \neq j. \end{cases}$$

Here d_i denotes the degree of node i. In the case of weighted graphs, we define $d_i = \sum_j A_{ij}$. So $L_G = D_G - A_G$, where D_G is the diagonal matrix of the degrees of G.

The (generally non-square) *incidence matrix* of G comes in two flavors. Let $V(G) = \{1, \dots, n\}$ and $E(G) = \{e_1, \dots, e_m$, and let B_G denote the $n \times m$ matrix for which

$$(B_G)_{ij} = \begin{cases} 1 & \text{if } i \text{ is and endpoint of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

Often, however, the following matrix is more useful: Let us fix an orientation of each edge, to get an oriented graph $\overrightarrow{G}$. Then let $B_{\overrightarrow{G}}$ denote the $V \times E$ matrix for which

$$(B_{\overrightarrow{G}})_{ij} = \begin{cases} 1 & \text{if } i = h(j), \\ -1 & \text{if } i = t(j), \\ 0 & \text{otherwise.} \end{cases}$$

Changing the orientation only means scaling some columns by -1, which often does not matter much. For example, it is easy to check that, independently of the orientation,

$$L_G = B_{\overrightarrow{G}} B_{\overrightarrow{G}}^{\mathsf{T}}. \tag{1.5}$$

It is worth while to express this equation in terms of quadratic forms:

$$x^{\mathsf{T}} L_G x = \sum_{ij \in E(G)}^{n} (x_i - x_j)^2. \tag{1.6}$$

The matrices A_G and L_G are symmetric, so their eigenvalues are real. Clearly $\sum_j L_{ij} = 0$, or in matrix form, $L\mathbb{1} = 0$, so L has a 0 eigenvalue. Equation (1.5) implies that L is positive semidefinite, so 0 is its smallest eigenvalue. The Perron–Frobenius Theorem implies that if

G is connected, then the largest eigenvalue $\lambda_{\max}$ of A_G has multiplicity 1. Applying the Perron–Frobenius Theorem to $cI - L_G$ with a sufficiently large scalar c, we see that for a connected graph, eigenvalue 0 of L_G has multiplicity 1.

Let $G = (V, E)$ be a graph. A *G-matrix* is any matrix $M \in \mathbb{R}^{V \times V}$ such that $M_{ij} = 0$ for every edge $ij \in \overline{E}$. A G-matrix is *well-signed*, if $M_{ij} < 0$ for all $ij \in E$. (Note that we have not imposed any condition on the diagonal entries.)

1.4 Convex bodies

1.4.1 Polytopes and polyhedra

The convex hull of a finite set of points in $\mathbb{R}^d$ is called a (convex) *polytope*. The intersection of a finite number of halfspaces in $\mathbb{R}^d$ is called a (convex) *polyhedron*. (We'll drop the adjective "convex", because we never need to talk about non-convex polyhedra.)

Proposition 1.4.1 *Every polytope is a polyhedron. A polyhedron is a polytope if and only if it is bounded.*

For every polyhedron, there is a unique smallest affine subspace that contains it, called its *affine hull*. The *dimension* of a polyhedron is the dimension of its affine hull. A polyhedron [polytope] in $\mathbb{R}^d$ that has dimension d (equivalently, that has an interior point) is called a *d-polyhedron* [*d-polytope*].

A hyperplane H is said to *support* the polytope if it has a point in common with the polytope and the polytope is contained in one of the closed halfspaces with boundary H. A *face* of a polytope is its intersection with a supporting hyperplane. A face of a polytope that has dimension one less than the dimension of the polytope is called a *facet*. A face of dimension 0 (i.e., a single point) is called a *vertex*.

Proposition 1.4.2 *Every face of a polytope is a polytope. Every vertex of a face is a vertex of the polytope. Every polytope has a finite number of faces.*

Every polytope is the convex hull of its vertices. The set of vertices is the unique minimal finite set of points whose convex hull is the polytope.

Every facet of a d-polyhedron P spans a (unique) hyperplane, and this hyperplane is the boundary of a uniquely determined halfspace that contains the polyhedron. The polyhedron is the intersection of the halfspaces determined by its facets this way.

1.4.2 Polar, blocker and antiblocker

Let P be a d-polytope containing the origin as an interior point. Then the *polar* of P is defined as

$$P^* = \{\mathbf{x} \in \mathbb{R}^d : \ \mathbf{x}^{\mathsf{T}}\mathbf{y} \leq 1 \ \forall \mathbf{y} \in P\}$$

Proposition 1.4.3 (a) *The polar of a polytope is a polytope. For every polytope P we have $(P^*)^* = P$.*

(b) *Let $\mathbf{v}_0, \ldots, \mathbf{v}_m$ be the vertices of a k-dimensional face F of P. Then*

$$F^\perp = \{\mathbf{x} \in P^* : \ \mathbf{v}_0^\mathsf{T}\mathbf{x} = 1, \ldots, \mathbf{v}_m^\mathsf{T}\mathbf{x} = 1\}$$

defines a $d-k-1$-dimensional face of P^*. Furthermore, $(F^\perp)^\perp = F$.

In particular, every vertex $\mathbf{v}$ of P corresponds to a facet $\mathbf{v}^\perp$ of P^* and vice versa. The vector $\mathbf{v}$ is a normal vector of the facet $\mathbf{v}^\perp$.

There are two constructions similar to polarity that concern polyhedra that do not contain the origin in their interior; rather, they are contained in the nonnegative orthant.

A polyhedron P in $\mathbb{R}^d$ is called *ascending*, if $P \subseteq \mathbb{R}^d_+$ and whenever $\mathbf{x} \in P$, $\mathbf{y} \in \mathbb{R}^d$ and $\mathbf{y} \geq \mathbf{x}$ then $\mathbf{y} \in P$.

The *blocker* of an ascending polyhedron is defined by

$$P^{\mathrm{bl}} = \{\mathbf{x} \in \mathbb{R}^d_+ : \ \mathbf{x}^\mathsf{T}\mathbf{y} \geq 1 \forall \mathbf{y} \in P\}.$$

Proposition 1.4.4 *The blocker of an ascending polyhedron is an ascending polyhedron. For every ascending polyhedron P we have $(P^{\mathrm{bl}})^{\mathrm{bl}} = P$.*

The correspondence between faces of P and P^{bl} is a bit more complicated than for polarity, and we describe the relationship between vertices and facets only. Every vertex $\mathbf{v}$ of P gives rise to a facet $\mathbf{v}\perp$, which determines the halfspace $\mathbf{v}^\mathsf{T}\mathbf{x} \geq 1$. This construction gives all the facets of P^{bl}, except possibly those corresponding to the nonnegativity constraints $x_i \geq 0$, which may or may not define facets.

A d-polytope P is called a *corner polytope*, if $P \subseteq \mathbb{R}^d_+$ and whenever $\mathbf{x} \in P$, $\mathbf{y} \in \mathbb{R}^d$ and $0 \leq \mathbf{y} \leq \mathbf{x}$ then $\mathbf{y} \in P$. The *antiblocker* of a corner polytope is defined by

$$P^{\mathrm{abl}} = \{\mathbf{x} \in \mathbb{R}^d_+ : \ \mathbf{x}^\mathsf{T}\mathbf{y} \leq 1 \forall \mathbf{y} \in P\}.$$

Proposition 1.4.5 *The antiblocker of a corner polytope is a corner polytope. For every corner polytope P we have $(P^{\mathrm{abl}})^{\mathrm{abl}} = P$.*

The correspondence between faces of P and P^{abl} is again a bit more complicated than for the polars. The nonnegativity constraints $x_i \geq 0$ always define facets, and they don't correspond to vertices in the antiblocker. All other facets of P correspond to vertices of P^{abl}. Not every vertex of P defines a facet in P^{abl}. The origin is a trivial exceptional vertex, but there may be further exceptional vertices. We call a vertex $\mathbf{v}$ *dominated*, if there is another vertex $\mathbf{w}$ such that $\mathbf{v} \leq \mathbf{w}$. Now a vertex of P defines a facet of P^* if and only if it is not dominated.

1.4.3 Volume

We denote by π_d the volume of the d-dimensional unit ball. It is known that

$$\pi_d = \frac{\pi^{d/2}}{\Gamma\left(1+\frac{d}{2}\right)} \sim \frac{(2e\pi)^{d/2}}{\sqrt{\pi} d^{(d+1)/2}}.$$

The volumes of a convex body and its polar are related. From the following bounds, we only use the upper bound (due to Blaschke and Santaló) in this book, but for completeness, we state the lower bound, due to Bourgain and Milman.

Proposition 1.4.6 *Let $K \subseteq \mathbb{R}^d$ be a 0-symmetric convex body. Then*

$$\frac{4^d}{d!} \leq \mathrm{vol}(K)\mathrm{vol}(K^*) \leq \pi_d^2.$$

The lower bound is attained when K is a cube and K^* is a cross-polytope (or the other way around). The upper bound is attained when both K and K^* are balls.

For a convex body K, the *difference body* $K-K$ is defined as the set of differences $\mathbf{x}-\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in K$. It is trivial that $\mathrm{vol}(K-K) \geq \mathrm{vol}(K)$ (since $K-K$ contains a translated copy of K). In fact, the Brunn-Minkowski Theorem implies that

$$\mathrm{vol}(K-K) \geq 2^d \mathrm{vol}(K). \tag{1.7}$$

1.4.4 Optimization

Let $P \subseteq \mathbb{R}^n$ be an ascending polyhedron. It is easy to see that P has a unique point which is closest to the origin. We'll see later that this point has combinatorial significance in some cases. Right now, we state the following simple theorem that relates this point for the analogous point in the blocker.

Theorem 1.4.7 *Let $P \subseteq \mathbb{R}^n$ be an ascending polyhedron, and let $\mathbf{x} \in P$ minimize the objective function $|\mathbf{x}|^2$. Let $\alpha = |\mathbf{x}|^2$ be the minimum value. Then $\mathbf{y} = (1/\alpha)\mathbf{x}$ is in the blocker P^{bl}, and it minimizes the objective function $|\mathbf{y}|^2$ over P^{bl}.*

1.5 Semidefinite optimization

Linear programming has been one of the most fundamental and successful tools in optimization and discrete mathematics. Linear programs are special cases of convex programs; *semidefinite programs* are more general but still convex programs, to which many of the useful properties of linear programs extend.

For more comprehensive studies of issues concerning semidefinite optimization, see [239, 156].

1.5.1 Semidefinite duality

A semidefinite program is an optimization problem of the following form:

$$\begin{aligned} &\text{minimize } c^{\mathsf{T}}x \\ &\text{subject to } x_1A_1 + \ldots x_nA_n - B \succeq 0 \end{aligned} \tag{1.8}$$

Here $A_1, \ldots, A_n, B$ are given symmetric $m \times m$ matrices, and $c \in \mathbb{R}^n$ is a given vector. We can think of $\sum_i x_iA_i - B$ as a matrix whose entries are linear functions of the variables.

As usual, any choice of the values x_i that satisfies the given constraint is called a *feasible solution*. A solution is *strictly feasible*, if the matrix $\sum_i x_iA_i - B$ is positive definite. We denote by v_{primal} the infimum of the objective function.

The special case when $A_1, \ldots, A_n, B$ are diagonal matrices is just a "generic" linear program, and it is very fruitful to think of semidefinite programs as generalizations of linear programs. But there are important technical differences. Unlike in the case of linear programs, the infimum may be finite but not a minimum, i.e., not attained by any feasible solution.

As in the theory of linear programs, there are a large number of equivalent formulations of a semidefinite program. Of course, we could consider minimization instead of maximization. We could allow additional linear constraints on the variables x_i (inequalities and/or equations). These could be incorporated into the form above by extending the A_i and B with new diagonal entries.

We could introduce the entries of the matrix $X = \sum_i x_iA_i - B$ as variables, in which case the fact that they are linear functions of the original variables translates into linear relations between them. Straightforward linear algebra transforms (1.8) into an optimization problem of the form

$$\begin{aligned} &\text{maximize } C \cdot X \\ &\text{subject to } X \succeq 0 \\ &\qquad\qquad D_i \cdot X = d_i \qquad (i = 1, \ldots, k) \end{aligned} \tag{1.9}$$

where $C, D_1, \ldots, D_k$ are symmetric $m \times m$ matrices and $d_1, \ldots, d_k \in \mathbb{R}$. Note that $C \cdot X$ is the general form of a linear combination of entries of X, and so $D_i \cdot X = d_i$ is the general form of a linear equation in the entries of X.

It is easy to see that we would not get any substantially more general problem if we allowed linear inequalities in the entries of X in addition to the equations.

The Farkas Lemma has a semidefinite version:

Lemma 1.5.1 (Homogeneous version) *Let $A_1, \ldots, A_n$ be symmetric $m \times m$ matrices. The system*

$$x_1A_1 + \cdots + x_nA_n \succ 0$$

has no solution in $x_1, \ldots, x_n$ if and only if there exists a symmetric matrix $Y \neq 0$ such that

$$\begin{aligned} A_i \cdot Y &= 0 \qquad (i = 1, \ldots, n) \\ Y &\succeq 0. \end{aligned}$$

There is also an inhomogeneous version of this lemma.

Lemma 1.5.2 (Inhomogeneous version) *Let $A_1, \ldots, A_n, B$ be symmetric $m \times m$ matrices. The system*

$$x_1 A_1 + \ldots x_n A_n - B \succ 0$$

has no solution in $x_1, \ldots, x_n$ if and only if there exists a symmetric matrix $Y \neq 0$ such that

$$\begin{aligned} A_i \cdot Y &= 0 \qquad (i = 1, \ldots, n) \\ B \cdot Y &\geq 0 \\ Y &\succeq 0. \end{aligned}$$

Given a semidefinite program (1.8), one can formulate the *dual program*:

$$\begin{aligned} &\text{maximize } B \cdot Y \\ &\text{subject to } A_i \cdot Y = c_i \qquad (i = 1, \ldots, n) \\ &\qquad\qquad\quad Y \succeq 0. \end{aligned} \tag{1.10}$$

Note that this too is a semidefinite program in the general sense. We denote by v_{dual} the infimum of the objective function.

With this notion of duality, the Duality Theorem holds under somewhat awkward conditions (which cannot be omitted; see e.g. [237, 231, 232, 188]):

Theorem 1.5.3 *Assume that both the primal program* (1.8) *and the dual program* (1.10) *have feasible solutions. Then $v_{\text{primal}} \leq v_{\text{dual}}$. If, in addition, the primal program (say) has a strictly feasible solution, then the dual optimum is attained and $v_{\text{primal}} = v_{\text{dual}}$.*

In particular, if both programs have strictly feasible solutions, then the supremum resp. infimum of the objective functions are attained and are equal. The following *complementary slackness conditions* also follow.

Proposition 1.5.4 *Let x be a feasible solution of the primal program* (1.8) *and Y, a feasible solution of the dual program* (1.10)*. Then $v_{\text{primal}} = v_{\text{dual}}$ and both x and Y are optimal solutions if and only if $Y(\sum_i x_i A_i - B) = 0$.*

1.5.2 Algorithms for semidefinite programs

There are two essentially different algorithms known that solve semidefinite programs in polynomial time: the *ellipsoid method* and *interior point/barrier methods*. Both of these have many variants, and the exact technical descriptions are quite complicated; we refer to [187, 188] for discussions of these.

The first polynomial time algorithm to solve semidefinite optimization problems in polynomial time was the ellipsoid method. This is based on the general fact that if we can test membership in a convex body $K \subseteq \mathbb{R}^d$ (i.e., we have a subroutine that, for a given point $x \in \mathbb{R}^d$, tells us whether or not $x \in K$), then we can use the ellipsoid method to optimize any linear objective function over K [95]. The precise statement of this fact needs nontrivial side-conditions.

For any semidefinite program (1.8), the set K of feasible solutions is convex. With rounding and other tricks, we can make it a bounded, full-dimensional set in $\mathbb{R}^n$. To test membership, we have to decide whether a given point x belongs to K; ignoring numerical problems, we can use Gaussian elimination to check whether the matrix $Y = \sum_i x_i A_i - B$ is positive semidefinite. Thus using the ellipsoid method we can compute, in polynomial time, a feasible solution x that is approximately optimal.

Unfortunately, the above argument gives an algorithm which is polynomial, but hopelessly slow, and practically useless. Semidefinite programs can be solved in polynomial time and also *practically efficiently* by interior point methods [183, 6, 7]. The algorithm can be described very informally as follows. We consider the form (1.9), denote by K the set of its feasible solutions (these are symmetric matrices), and want to minimize a linear function $C \cdot X$ over $X \in K$. The set K is convex, but the minimum will be attained on the boundary of K, and this boundary is neither smooth nor polyhedral in general. Therefore, neither gradient-type methods nor simplex-type methods of linear programming can be used.

The main idea of barrier methods is that instead of minimizing a linear objective function $C^{\mathsf{T}}X$, we minimize the convex function $F_\lambda(x) = -\log\det(X) + \lambda C^{\mathsf{T}}X$ for some parameter $\lambda > 0$. Since F_λ tends to infinity on the boundary of K, the minimum will be attained in the interior. Since F_λ is convex and analytic in the interior, the minimum can be very efficiently computed by a variety of numerical methods (conjugate gradient etc.)

Of course, the point we obtain this way is not what we want, but if λ is large it will be close. If we don't like it, we can increase λ and use the minimizing point for the old F_λ as the starting point for a new gradient type algorithm. One can show that (under some technical assumptions about the feasible domain) this algorithm gives a good approximation of the optimum in polynomial time (see [7, 231, 232] and the book [182]).

Bibliography

[1] B.M. Ábrego, S. Fernández-Merchant: A lower bound for the rectilinear crossing number. Manuscript, 2003.

[2] O. Aichholzer, F. Aurenhammer and H. Krasser: On the crossing number of complete graphs, in: *Proc.* 18^{th} *Ann. ACM Symp. Comp. Geom.* (2002), pp. 19–24.

[3] M. Ajtai, Chvátal, V.; Newborn, M.; Szemerédi, E. (1982): Crossing-free subgraphs, *Theory and Practice of Combinatorics*, North-Holland Mathematics Studies **60** 9–12.

[4] M. Ajtai, J. Komlós and E. Szemerédi: A note on Ramsey numbers, *J. Combin. Theory A* 29 (1980), 354–360.

[5] D. J. Aldous and J. Fill: *Reversible Markov Chains and Random Walks on Graphs* (book in preparation); URL for draft at http://www.stat.Berkeley.edu/users/aldous

[6] F. Alizadeh: Combinatorial optimization with semi-definite matrices, in: *Integer Programming and Combinatorial Optimization* (Proceedings of IPCO '92), (eds. E. Balas, G. Cornuéjols and R. Kannan), Carnegie Mellon University Printing (1992), 385–405.

[7] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization, *SIAM J. Optim.* **5** (1995), 13–51.

[8] F. Alizadeh, J.-P. Haeberly, and M. Overton: Complementarity and nondegeneracy in semidefinite programming, in: *Semidefinite Programming*, *Math. Programming* Ser. B, **77** (1997), 111–128.

[9] N. Alon: Eigenvalues and expanders, *Combinatorica* **6**(1986), 83–96.

[10] N. Alon: The Shannon capacity of a union, *Combinatorica* **18** (1998), 301–310.

[11] N. Alon and E. Lubetzky: The Shannon capacity of a graph and the independence numbers of its powers, *IEEE Trans. Inform. Theory* **52** (2006), 2172–2176.

[12] N. Alon and E. Győri: The number of small semispaces of a finite set of points in the plane, *J. Combin. Theory Ser. A* **41** (1986), 154–157.

[13] N. Alon and N. Kahale: Approximating the independence number via the ϑ-function, *Math. Programming* **80** (1998), Ser. A, 253–264.

[14] N. Alon and V. D. Milman: λ_1, isoperimetric inequalities for graphs and superconcentrators, *J. Combinatorial Theory B* **38**(1985), 73–88.

[15] N. Alon, L. Lovász: Unextendible product bases, *J. Combin. Theory Ser. A* **95** (2001), 169–179.

[16] N. Alon, P.D. Seymour and R. Thomas: Planar separators, *SIAM J. Disc. Math.* **7** (1994), 184–193.

[17] N. Alon and J.H. Spencer: *The Probabilistic Method*, Wiley, New York, 1992.

[18] B. Andrásfai: On critical graphs, *Theory of Graphs (International Symposium, Rome, 1966)*, ed. P. Rosenstiehl, Gordon and Breach, New York (1967), 9–19.

[19] E. Andre'ev: On convex polyhedra in Lobachevsky spaces, *Mat. Sbornik*, Nov. Ser. **81** (1970), 445–478.

[20] E. Andre'ev: On convex polyhedra of finite volume in Lobachevsky spaces, *Mat. Sbornik*, Nov. Ser. **83** (1970), 413–440.

[21] S. Arora, C. Lund, R. Motwani, M. Sudan, M. Szegedy: Proof verification and hardness of approximation problems *Proc. 33rd FOCS* (1992), 14–23.

[22] L. Asimov and B. Roth: The rigidity of graphs, *Trans. Amer. Math. Soc.*, **245** (1978), 279–289.

[23] R. Bacher and Y. Colin de Verdière: Multiplicités des valeurs propres et transformations étoile-triangle des graphes, *Bull. Soc. Math. France* **123** (1995), 101-117.

[24] E.G. Bajmóczy, I. Bárány, On a common generalization of Borsuk's and Radon's theorem, *Acta Mathematica Academiae Scientiarum Hungaricae* 34 (1979) 347–350.

[25] A.I. Barvinok: Problems of distance geometry and convex properties of quadratic maps, *Discrete and Comp. Geometry* **13** (1995), 189–202.

[26] A.I. Barvinok: A remark on the rank of positive semidefinite matrices subject to affine constraints, *Discrete and Comp. Geometry* (to appear)

[27] M. Bellare, O. Goldreich, and M. Sudan: Free bits, PCP and non-approximability—towards tight results, *SIAM J. on Computing* **27** (1998), 804–915.

[28] I. Benjamini and L. Lovász: Global Information from Local Observation, *Proc. 43rd Ann. Symp. on Found. of Comp. Sci.* (2002) 701–710.

[29] I. Benjamini and L. Lovász: Harmonic and analytic frunctions on graphs, *Journal of Geometry* **76** (2003), 3–15.

[30] I. Benjamini and O. Schramm: Harmonic functions on planar and almost planar graphs and manifolds, via circle packings. *Invent. Math.* **126** (1996), 565–587.

[31] I. Benjamini and O. Schramm: Random walks and harmonic functions on infinite planar graphs using square tilings, *Ann. Probab.* **24** (1996), 1219–1238.

[32] C. Berge: *Graphs and Hypergraphs*, North-Holland, Amsterdam (1973).

[33] T. Böhme, On spatial representations of graphs, in: *Contemporary Methods in Graph Theory* (R. Bodendieck, ed.), BI-Wiss.-Verl. Mannheim, Wien/Zurich (1990), 151–167.

[34] J. Bourgain: On Lipschitz embedding of finite metric spaces in Hilbert space, *Israel J. Math.* **52** (1985), 46–52.

[35] P. Brass: On the maximum number of unit distances among n points in dimension four, *Intuitive geometry* Bolyai Soc. Math. Stud. **6**, J. Bolyai Math. Soc., Budapest (1997), 277–290.

[36] P. Brass: Erdős distance problems in normed spaces, *Comput. Geom.* **6** (1996), 195–214.

[37] P. Brass: The maximum number of second smallest distances in finite planar sets. *Discrete Comput. Geom.* **7** (1992), 371–379.

[38] P. Brass, G. Károlyi: On an extremal theory of convex geometric graphs (manuscript)

[39] G.R. Brightwell, E. Scheinerman: Representations of planar graphs, *SIAM J. Discrete Math.* **6** (1993) 214–229.

[40] R.L. Brooks, C.A.B. Smith, A.H. Stone and W.T. Tutte, The dissection of rectangles into squares. Duke Math. J. 7, (1940). 312–340.

[41] A.E. Brouwer and W.H. Haemers: Association Schemes, in *Handbook of Combinatorics* (ed. R. Graham, M. Grötschel, L. Lovász), Elsevier Science B.V. (1995), 747–771.

[42] P. Cameron, J.-M. Goethals, J.J. Seidel and E.E. Shult, Line graphs, root systems and elliptic geometry, *J. Algebra* **43** (1976), 305–327.

[43] A.K. Chandra, P. Raghavan, W.L. Ruzzo, R. Smolensky and P. Tiwari: The electrical resistance of a graph captures its commute and cover times, *Proc. 21st ACM STOC* (1989), 574–586.

[44] S.Y. Cheng, Eigenfunctions and nodal sets, *Commentarii Mathematici Helvetici* 51 (1976) 43–55.

[45] J. Cheriyan, J.H. Reif: Directed s-t numberings, rubber bands, and testing digraph k-vertex connectivity, *Proceedings of the Third Annual ACM-SIAM Symposium on Discrete Algorithms*, ACM, New York (1992), 335–344.

[46] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas: The strong perfect graph theorem, *Annals of Math.*, **164** (2006), 51–229.

[47] Y. Colin de Verdière: Sur la multiplicité de la première valeur propre non nulle du laplacien, *Comment. Math. Helv.* **61** (1986), 254–270.

[48] Y. Colin de Verdière: Sur un nouvel invariant des graphes et un critère de planarité, *Journal of Combinatorial Theory, Series B* **50** (1990), 11–21. English translation: On a new graph invariant and a criterion for planarity, in: *Graph Structure Theory* (Robertson and P. D. Seymour, eds.), Contemporary Mathematics, Amer. Math. Soc., Providence, RI (1993), 137–147.

[49] Y. Colin de Verdière: Une principe variationnel pour les empilements de cercles, *Inventiones Math.* **104** (1991) 655–669.

[50] Y. Colin de Verdière: Multiplicities of eigenvalues and tree-width of graphs, *J. Combin. Theory Ser. B* **74** (1998), 121–146.

[51] Y. Colin de Verdière: *Spectres de graphes*, Cours Spécialisés **4**, Société Mathématique de France, Paris, 1998.

[52] R. Connelly: A counterexample to the rigidity conjecture for polyhedra, *Publications Mathématiques de l'IHÉS* **47** (1977), 333–338.

[53] G. Csizmadia: The multiplicity of the two smallest distances among points, *Discrete Math.* **194** (1999), 67–86.

[54] A. Coja-Oghlan: The Lovász number of random graphs, in: *Approximation, randomization, and combinatorial optimization*, Lecture Notes in Comput. Sci. **2764**, Springer, Berlin (2003), 228–239.

[55] A. Coja-Oghlan and A. Taraz: Exact and approximative algorithms for coloring $G(n,p)$, *Random Structures and Algorithms* **24** (2004), 259–278.

[56] C. Delorme and S. Poljak: Laplacian eigenvalues and the maximum cut problem, *Math. Programming* **62** (1993) 557–574.

[57] C. Delorme and S. Poljak: Combinatorial properties and the complexity of max-cut approximations, *Europ. J. Combin.* **14** (1993), 313–333.

[58] T.L. Dey: Improved bounds for planar k-sets and related problems, *Discrete and Computational Geometry* **19** (1998), 373–382.

[59] M. Deza and M. Laurent: *Geometry of Cuts and Metrics*, Springer Verlag, 1997.

[60] P.D. Doyle and J.L. Snell: *Random walks and electric networks,* Math. Assoc. of Amer., Washington, D.C. 1984.

[61] R.J. Duffin: Basic properties of discrete analytic functions, *Duke Math. J.* **23** (1956), 335–363.

[62] R.J. Duffin: Potential theory on the rhombic lattice, *J. Comb. Theory* **5** (1968) 258–272.

[63] R.J. Duffin and E.L. Peterson: The discrete analogue of a class of entire functions, *J. Math. Anal. Appl.* **21** (1968) 619–642.

[64] I.A. Dynnikov and S.P. Novikov: Geometry of the Triangle Equation on Two-Manifolds, *Mosc. Math. J.* **3** (2003), 419–438.

[65] H. Edelsbrunner, P. Hajnal: A lower bound on the number of unit distances between the vertices of a convex polygon, *J. Combin. Theory Ser. A* **56** (1991), 312–316.

[66] J. Edmonds: Submodular functions, matroids, and certain polyhedra, in: *Combinatorial structures and their applications* (eds. R.K. Guy, H. Hanani, N. Sauer and J. Schonheim), Gordon and Breach, New York (1970), pp. 69–87.

[67] H.G. Eggleston: Covering a three-dimensional set with sets of smaller diameter, *J. London Math. Soc.* **30** (1955), 11–24.

[68] P. Erdős, L. Lovász, G. Simmons, and E. Strauss: Dissection graphs of planar point sets, in: *A Survey of Combinatorial Theory*, North Holland, Amsterdam, (1973), pp. 139–149.

[69] P. Erdős, On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250.

[70] P. Erdős: On sets of distances of n points in Euclidean space, *Publ. Math. Inst. Hung. Acad. Sci.* **5** (1960), 165–169.

[71] P. Erdős: Gráfok páros körüljárású részgráfjairól (On bipartite subgraphs of graphs, in Hungarian), *Mat. Lapok* **18** (1967), 283–288.

[72] P. Erdős: On some applications of graph theory to geometry, *Canad. J. Math.* **19** (1967), 968–971.

[73] P. Erdős and N.G. de Bruijn: A colour problem for infinite graphs and a problem in the theory of relations, *Nederl. Akad. Wetensch. Proc. – Indagationes Math.* **13** (1951), 369–373.

[74] P. Erdős and T. Gallai: On the minimal number of vertices representing the edges of a graph, *Magyar Tud. Akad. Mat. Kut. Int. Közl* **6** (1961), 181–203.

[75] P. Erdős, F. Harary and W.T. Tutte, On the dimension of a graph *Mathematika* **12** (1965), 118–122.

[76] P. Erdős, L. Lovász, K. Vesztergombi: On the graph of large distances, *Discrete and Computational Geometry* **4** (1989) 541–549.

[77] P. Erdős, L. Lovász, K. Vesztergombi: The chromatic number of the graph of large distances, in: *Combinatorics*, Proc. Coll. Eger 1987, Coll. Math. Soc. J. Bolyai **52**, North-Holland, 547–551.

[78] P. Erdős and M. Simonovits: On the chromatic number of geometric graphs, *Ars Combin.* **9** (1980), 229–246.

[79] U. Feige: Approximating the bandwidth via volume respecting embeddings, Proc. 30th Ann. ACM Symp. on Theory of Computing, ACM New York, NY (1998), 90-99.

[80] Z. Fredi and J. Komls: The eigenvalues of random symmetric matrices, *Combinatorica* **1** (1981), 233–241.

[81] C. Dubey, U. Feige, W. Unger: Hardness results for approximating the bandwidth, *Journal of Computer and System Sciences* **77** (2010), 62–90.

[82] P. Erdős, A. Hajnal and J. Moon: A problem in graph theory, *Amer. Math. Monthly* **71** (1964), 1107–1110.

[83] J. Ferrand, Fonctions préharmoniques et fonctions préholomorphes. (French) *Bull. Sci. Math.* **68**, (1944). 152–180.

[84] C.M. Fiduccia, E.R. Scheinerman, A. Trenk, J.S. Zito: Dot product representations of graphs, *Discrete Math.* **181** (1998), 113–138.

[85] H. de Fraysseix, J. Pach, R. Pollack: How to draw a planar graph on a grid, *Combinatorica* **10** (1990), 41–51.

[86] P. Frankl, H. Maehara: The Johnson-Lindenstrauss lemma and the sphericity of some graphs, *J. Combin. Theory Ser. B* **44** (1988), 355–362.

[87] P. Frankl, H. Maehara: On the contact dimension of graphs, *Disc. Comput. Geom.* **3** (1988) 89–96.

[88] P. Frankl, R.M. Wilson: Intersection theorems with geometric consequences, *Combinatorica* **1** (1981), 357–368.

[89] S. Friedland and R. Loewy, Subspaces of symmetric matrices containing matrices with multiple first eigenvalue, *Pacific J. Math.* **62** (1976), 389–399.

[90] Z. Füredi: The maximum number of unit distances in a convex n-gon, *J. Combin. Theory Ser. A* **55** (1990), 316–320.

[91] M. X. Goemans and D. P. Williamson: Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming, *J. Assoc. Comput. Mach.* **42** (1995), 1115–1145.

[92] M. Grötschel, L. Lovász and A. Schrijver: The ellipsoid method and its consequences in combinatorial optimization, *Combinatorica* **1** (1981), 169-197.

[93] M. Grötschel, L. Lovász and A. Schrijver: Polynomial algorithms for perfect graphs, *Annals of Discrete Math.* **21** (1984), 325-256.

[94] M. Grötschel, L. Lovász, A. Schrijver: Relaxations of vertex packing, *J. Combin. Theory B* **40** (1986), 330-343.

[95] M. Grötschel, L. Lovász, A. Schrijver: Geometric Algorithms and Combinatorial Optimization, Springer (1988).

[96] B. Grünbaum: A proof of Vázsonyi's conjecture, *Bull. Res. Council Israel, section A* **6** (1956), 77–78.

[97] B. Grünbaum, T.S. Motzkin: On polyhedral graphs, *Proc. Sympos. Pure Math.* **7** (1963), Amer. Math. Soc., Providence, R.I., 285–290.

[98] Richard K. Guy: A combinatorial problem, *Bull. Malayan Math. Soc.* **7** (1967), 68–72.

[99] H. Hadwiger: Ein Überdeckungssatz für den Euklidischen Raum, *Portugaliae Math. 4* (1944), 140–144.

[100] W. Haemers: On some problems of Lovász concerning the Shannon capacity of a graph, *IEEE Trans. Inform. Theory* **25** (1979), 231–232.

[101] J. Håstad: Some optimal in-approximability results, *Proc. 29th ACM Symp. on Theory of Comp.*, 1997, 1–10.

[102] J. Håstad: Clique is hard to approximate within a factor of $n^{1-\varepsilon}$, *Acta Math.* **182** (1999), 105–142.

[103] Z.-X. He, O. Schramm: Hyperbolic and parabolic packings, *Discrete Comput. Geom.* **14** (1995), 123–149.

[104] Z.-X. He, O. Schramm: On the convergence of circle packings to the Riemann map, *Invent. Math.* **125** (1996), 285–305.

[105] A. Heppes: Beweis einer Vermutung von A. Vázsonyi, *Acta Math. Acad. Sci. Hung.* **7** (1956), 463–466.

[106] A. Heppes, P. Révész: A splitting problem of Borsuk. (Hungarian) *Mat. Lapok* **7** (1956), 108–111.

[107] A.J. Hoffman, Some recent results on spectral properties of graphs, in: *Beiträge zur Graphentheorie,* Teubner, Leipzig (1968) 75–80.

[108] H. van der Holst: A short proof of the planarity characterization of Colin de Verdière, *Journal of Combinatorial Theory, Series B* 65 (1995) 269–272.

[109] H. van der Holst: *Topological and Spectral Graph Characterizations*, Ph.D. Thesis, University of Amsterdam, Amsterdam, 1996.

[110] H. van der Holst, M. Laurent, A. Schrijver, On a minor-monotone graph invariant, *Journal of Combinatorial Theory, Series B* 65 (1995) 291–304.

[111] H. van der Holst, L. Lovász and A. Schrijver: On the invariance of Colin de Verdière's graph parameter under clique sums, *Linear Algebra and its Applications*, **226–228** (1995), 509–518.

[112] H. van der Holst, L. Lovász, A. Schrijver: The Colin de Verdière graph parameter, in: *Graph Theory and Combinatorial Biology*, Bolyai Soc. Math. Stud. **7**, János Bolyai Math. Soc., Budapest (1999), 29–85.

[113] A.J. Hoffman: On graphs whose least eigenvalue exceeds $-1-\sqrt{2}$, *Linear Algebra and Appl.* **16** (1977), 153–165.

[114] A.J. Hoffman: On eigenvalues and colorings of graphs. *Graph Theory and its Applications*, Academic Press, New York (1970), 79–91.

[115] H. van der Holst, L. Lovász, A. Schrijver: The Colin de Verdière graph parameter, in: *Graph Theory and Combinatorial Biology*, Bolyai Soc. Math. Stud. **7**, János Bolyai Math. Soc., Budapest (1999), 29–85.

[116] H. Hopf, E. Pannwitz: *Jahresbericht der Deutsch. Math. Vereinigung* **43** (1934), 2 Abt. **114**, Problem 167.

[117] M. Iri: On an extension of the maximum-flow minimum cut theorem to multicommodity flows, *J. Oper. Res. Soc. Japan* **5** (1967), 697–703.

[118] R. Isaacs, Monodiffric functions, *Natl. Bureau Standards App. Math. Series* **18** (1952), 257–266.

[119] W.B. Johnson, J. Lindenstrauss: Extensions of Lipschitz mappings into a Hilbert space. *Conference in modern analysis and probability* (New Haven, Conn., 1982), 189–206, Contemp. Math., 26, Amer. Math. Soc., Providence, R.I., 1984.

[120] M.R. Jerrum and A. Sinclair: Approximating the permanent, *SIAM J. Comput.* **18**(1989), 1149–1178.

[121] J. Jonasson and O. Schramm: On the cover time of planar graphs *Electron. Comm. Probab.* **5** (2000), paper no. 10, 85–90.

[122] F. Juhász: The asymptotic behaviour of Lovász' ϑ function for random graphs, *Combinatorica* **2** (1982) 153–155.

[123] G.A. Kabatjanskiĭ and V.I. Levenštein: Bounds for packings on a sphere and in space (in Russian), *Problemy Peredachi Informatsii* **14** (1978), 3–25.

[124] J. Kahn, G, Kalai: A counterexample to Borsuk's conjecture. *Bull. Amer. Math. Soc.* **29** (1993), 60–62.

[125] R. Kang, L. Lovász, T. Muller and E. Scheinerman: Dot product representations of planar graphs, *Electr. J. Combin.* **18** (2011), P216.

[126] D. Karger, R. Motwani, M. Sudan: Approximate graph coloring by semidefinite programming, *Proc. 35th FOCS* (1994), 2–13.

[127] B. S. Kashin and S. V. Konyagin: On systems of vectors in Hilbert spaces, *Trudy Mat. Inst. V.A.Steklova* **157** (1981), 64–67; English translation: *Proc. of the Steklov Inst. of Math.* (AMS 1983), 67–70.

[128] R. Kenyon: The Laplacian and Dirac operators on critical planar graphs, *Inventiones Math* **150** (2002), 409-439.

[129] R. Kenyon and J.-M. Schlenker: Rhombic embeddings of planar graphs with faces of degree 4 (math-ph/0305057).

[130] S. Khot, G. Kindler, E. Mossel and R. O'Donnell: Optimal inapproximability results for MAX-CUT and other two-variable CSP's, *SIAM Journal on Computing* **37**, 319–357.

[131] D. E. Knuth: The sandwich theorem, *The Electronic Journal of Combinatorics* **1** (1994), 48 pp.

[132] P. Koebe: Kontaktprobleme der konformen Abbildung, *Berichte über die Verhandlungen d. Sächs. Akad. d. Wiss.*, Math.–Phys. Klasse, **88** (1936) 141–164.

[133] V. S. Konyagin, Systems of vectors in Euclidean space and an extremal problem for polynomials, *Mat. Zametky* **29** (1981), 63–74. English translation: *Math. Notes of the Academy USSR* **29** (1981), 33–39.

[134] A. Kotlov: A spectral characterization of tree-width-two graphs, *Combinatorica* **20** (2000)

[135] A. Kotlov, L. Lovász: The rank and size of graphs, *J. Graph Theory* **23** (1996), 185–189.

[136] A. Kotlov, L. Lovász, S. Vempala, The Colin de Verdière number and sphere representations of a graph, *Combinatorica* **17** (1997) 483–521.

[137] G. Laman (1970): On graphs and rigidity of plane skeletal structures, *J. Engrg. Math.* **4**, 331–340.

[138] P. Lancaster, M. Tismenetsky, *The Theory of Matrices, Second Edition, with Applications*, Academic Press, Orlando, 1985.

[139] D.G. Larman, C.A. Rogers: The realization of distances within sets in Euclidean space. *Mathematika* **19** (1972), 1–24.

[140] M. Laurent and S. Poljak: On the facial structure of the set of correlation matrices, *SIAM J. on Matrix Analysis and Applications* **17** (1996), 530–547.

[141] F.T. Leighton: *Complexity Issues in VLSI*, Foundations of Computing Series, MIT Press, Cambridge, MA (1983).

[142] F.T. Leighton and S. Rao, An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms, *Proc. 29th Annual Symp. on Found. of Computer Science,* IEEE Computer Soc., (1988), 422-431.

[143] N. Linial, E. London, Y. Rabinovich: The geometry of graphs and some of its algorithmic applications, *Combinatorica* **15** (1995), 215–245.

[144] N. Linial, L. Lovász, A. Wigderson: Rubber bands, convex embeddings, and graph connectivity, *Combinatorica* **8** (1988) 91-102.

[145] R.J. Lipton, R.E. Tarjan: A Separator Theorem for Planar Graphs, *SIAM J. on Applied Math.*, **36** (1979), 177–189.

[146] L. Lovász: Normal hypergraphs and the perfect graph conjecture, *Discrete Math.* **2** (1972), 253-267.

[147] L. Lovász: Combinatorial Problems and Exercises, Second Edition, Akadémiai Kiadó - North Holland, Budapest, 1991

[148] L. Lovász: Flats in matroids and geometric graphs, in: *Combinatorial Surveys*, Proc. 6th British Comb. Conf., Academic Press (1977), 45–86.

[149] L. Lovász: Some finite basis theorems in graph theory, in: *Combinatorics*, Coll. Math. Soc. J. Bolyai **18** (1978), 717–729.

[150] L. Lovász: On the Shannon capacity of graphs, *IEEE Trans. Inform. Theory* **25** (1979), 1–7.

[151] L. Lovász: Selecting independent lines from a family of lines in a space, **Acta Sci. Math. Szeged 42** (1980), 121–131.

[152] L. Lovász: Submodular functions and convexity, in: *Mathematical Programming: the State of the Art* (ed. A.Bachem, M.Grötschel, B.Korte), Springer (1983), 235–257.

[153] L. Lovász: Singular spaces of matrices and their applications in combinatorics, *Bol. Soc. Braz. Mat.* **20** (1989), 87–99.

[154] L. Lovász: Random walks on graphs: a survey, in: *Combinatorics, Paul Erdős is Eighty*, Vol. 2 (ed. D. Miklos, V. T. Sos, T. Szonyi), János Bolyai Mathematical Society, Budapest, 1996, 353–398.

[155] L. Lovász: Steinitz representations of polyhedra and the Colin de Verdière number, *J. Comb. Theory B* **82** (2001), 223–236.

[156] L. Lovász: Semidefinite programs and combinatorial optimization, in: *Recent Advances in Algorithms and Combinatorics,* CMS Books Math./Ouvrages Math. SMC *11*, Springer, New York (2003), 137–194.

[157] L. Lovász: Graph minor theory, *Bull. Amer. Math. Soc.* **43** (2006), 75–86.

[158] L. Lovász: *Large graphs, graph homomorphisms and graph limits*, Amer. Math. Soc., Providence, RI (2012).

[159] L. Lovász, M.D. Plummer: *Matching Theory*, Akadémiai Kiadó–North Holland, Budapest, 1986 (reprinted by AMS Chelsea Publishing, 2009).

[160] L. Lovász, M. Saks, A. Schrijver: Orthogonal representations and connectivity of graphs, *Linear Alg. Appl.* **114/115** (1989), 439–454; A Correction: Orthogonal representations and connectivity of graphs, *Linear Alg. Appl.* **313** (2000) 101–105.

[161] L. Lovász, A. Schrijver: A Borsuk theorem for antipodal links and a spectral characterization of linklessly embedable graphs, *Proceedings of the American Mathematical Society* **126** (1998) 1275–1285.

[162] L. Lovász, A. Schrijver: On the null space of a Colin de Verdière matrix, *Annales de l'Institute Fourier* **49**, 1017–1026.

[163] L. Lovász and B. Szegedy: Szemerédi's Lemma for the analyst, *Geom. Func. Anal.* **17** (2007), 252–270.

[164] L. Lovász, K. Vesztergombi: Geometric representations of graphs, to appear in *Paul Erdos and his Mathematics,* (ed. G. Halász, L. Lovász, M. Simonovits, V.T. Sós), Bolyai Society–Springer Verlag.

[165] L. Lovász, K. Vesztergombi, U. Wagner, E. Welzl: Convex quadrilaterals and k-sets, in: *Towards a Theory of Geometric Graphs*, (J. Pach, Ed.), AMS Contemporary Mathematics **342** (2004), 139–148.

[166] L. Lovász and P. Winkler: Mixing times, in: *Microsurveys in Discrete Probability* (ed. D. Aldous and J. Propp), DIMACS Series in Discr. Math. and Theor. Comp. Sci., **41** (1998), 85–133.

[167] L. Lovász, Y. Yemini: On generic rigidity in the plane, *SIAM J. Alg. Discr. Methods* **1** (1982), 91-98.

[168] Lubotzky, Phillips, Sarnak

[169] H. Maehara, V. Rödl: On the dimension to represent a graph by a unit distance graph, *Graphs and Combinatorics* **6** (1990), 365–367.

[170] P. Mani, Automorphismen von polyedrischen Graphen, *Math. Annalen* **192** (1971), 279–303.

[171] Margulis [expander]

[172] J. Matoušek: On embedding expanders into l^p spaces. *Israel J. Math.* **102** (1997), 189–197.

[173] C. Mercat, Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.* **218** (2001), no. 1, 177–216.

[174] C. Mercat: Discrete polynomials and discrete holomorphic approximation (2002)

[175] C. Mercat: Exponentials form a basis for discrete holomorphic functions (2002)

[176] G.L. Miller, W. Thurston: Separators in two and three dimensions, *Proc. 22th ACM STOC* (1990), 300–309.

[177] G.L. Miller, S.-H. Teng, W. Thurston, S.A. Vavasis: Separators for sphere-packings and nearest neighbor graphs. *J. ACM* **44** (1997), 1–29.

[178] B. Mohar and S. Poljak: Eigenvalues and the max-cut problem, *Czechoslovak Mathematical Journal* **40** (1990), 343–352.

[179] B. Mohar and C. Thomassen: Graphs on surfaces. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

[180] A. Mowshowitz: Graphs, groups and matrices, in: *Proc. 25th Summer Meeting Canad. Math. Congress, Congr. Numer.* **4** (1971), Utilitas Mathematica, Winnipeg, 509–522.

[181] C. St.J. A. Nash-Williams, Random walks and electric currents in networks, *Proc. Cambridge Phil. Soc.* **55** (1959), 181–194.

[182] Yu. E. Nesterov and A. Nemirovsky: *Interior-point polynomial methods in convex programming,* Studies in Appl. Math. **13**, SIAM, Philadelphia, 1994.

[183] M. L. Overton: On minimizing the maximum eigenvalue of a symmetric matrix, *SIAM J. on Matrix Analysis and Appl.* **9** (1988), 256–268.

[184] M. L. Overton and R. Womersley: On the sum of the largest eigenvalues of a symmetric matrix, *SIAM J. on Matrix Analysis and Appl.* **13** (1992), 41–45.

[185] G. Pataki: On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues, *Math. of Oper. Res.* **23** (1998), 339–358.

[186] S. Poljak and F. Rendl: Nonpolyhedral relaxations of graph-bisection problems, DIMACS Tech. Report 92-55 (1992).

[187] L. Porkoláb and L. Khachiyan: On the complexity of semidefinite programs, *J. Global Optim.* **10** (1997), 351–365.

[188] M. Ramana: An exact duality theory for semidefinite programming and its complexity implications, in: *Semidefinite programming. Math. Programming* Ser. B, **77** (1997), 129–162.

[189] A. Recski: *Matroid theory and its applications in electric network theory and in statics*, Algorithms and Combinatorics, 6. Springer-Verlag, Berlin; Akadémiai Kiadó, Budapest, 1989.

[190] J. Reiterman, V. Rödl, E. Šinajová: Embeddings of graphs in Euclidean spaces, *Discr. Comput. Geom.* **4** (1989), 349–364.

[191] J. Reiterman, V. Rödl and E. Šinajová: On embedding of graphs into Euclidean spaces of small dimension, *J. Combin. Theory Ser. B* **56** (1992), 1–8.

[192] J. Richter-Gebert: *Realization spaces of polytopes*, Lecture Notes in Mathematics **1643**, Springer-Verlag, Berlin, 1996.

[193] N. Robertson, P.D. Seymour: Graph minors III: Planar tree-width, *J. Combin. Theory B* **36** (1984), 49–64.

[194] N. Robertson, P.D. Seymour, Graph minors XX: Wagner's conjecture (preprint, 1988).

[195] N. Robertson, P. Seymour, R. Thomas: A survey of linkless embeddings, in: *Graph Structure Theory* (N. Robertson, P. Seymour, eds.), Contemporary Mathematics, American Mathematical Society, Providence, Rhode Island, 1993, pp. 125–136.

[196] N. Robertson, P. Seymour, R. Thomas: Hadwiger's conjecture for K_6-free graphs, *Combinatorica* **13** (1993) 279–361.

[197] N. Robertson, P. Seymour, R. Thomas: Sachs' linkless embedding conjecture, *Journal of Combinatorial Theory, Series B* **64** (1995) 185–227.

[198] B. Rodin and D. Sullivan: The convergence of circle packings to the Riemann mapping, *J. Diff. Geom.* **26** (1987), 349–360.

[199] H. Sachs,

[200] H. Sachs, Coin graphs, polyhedra, and conformal mapping, *Discrete Math.* **134** (1994), 133–138.

[201] F. Saliola and W. Whiteley: Constraining plane configurations in CAD: Circles, lines and angles in the plane *SIAM J. Discrete Math.* **18** (2004), 246–271.

[202] H. Saran: *Constructive Results in Graph Minors: Linkless Embeddings*, Ph.D. Thesis, University of California at Berkeley, 1989.

[203] W. Schnyder: Embedding planar graphs on the grid, *Proc. First Annual ACM-SIAM Symp. on Discrete Algorithms*, SIAM, Philadelphia (1990), 138–148;

[204] O. Schramm: Existence and uniqueness of packings with specified combinatorics, *Israel Jour. Math.* **73** (1991), 321–341.

[205] O. Schramm: How to cage an egg, *Invent. Math.* **107** (1992), 543–560.

[206] O. Schramm: Square tilings with prescribed combinatorics, *Israel Jour. Math.* **84** (1993), 97–118.

[207] A. Schrijver, Minor-monotone graph invariants, in: *Combinatorics 1997* (P. Cameron, ed.), Cambridge University Press, Cambridge, 1997, to appear.

[208] J. Schwartz: Fast probabilistic algorithms for verification of polynomial identities, *Journal of the ACM* **27** (1980), 701–717.

[209] F. Shahroki and D.W. Matula: The maximum concurrent flow problem, J. ACM **37** (1990), 318–334.

[210] D. Singer: Rectilinear crossing numbers, Manuscript (1971), available at `http://www.cwru.edu/artsci/math/singer/home.html`.

[211] P.M. Soardi: *Potential Theory on Infinite Networks*, Lecture notes in Math. **1590**, Springer-Verlag, Berlin–Heidelberg, 1994.

[212] J. Spencer, E. Szemerédi and W.T. Trotter: Unit distances in the Euclidean plane, *Graph Theory and Combinatorics*, Academic Press, London–New York (1984), 293–303.

[213] D.A. Spielman and S.-H. Teng: Disk Packings and Planar Separators, *12th Annual ACM Symposium on Computational Geometry* (1996), 349–358.

[214] D.A. Spielman and S.-H. Teng: Spectral Partitioning Works: Planar graphs and finite element meshes, *Proceedings of the 37th Annual IEEE Conference on Foundations of Comp. Sci.* (1996), 96–105.

[215] E. Steinitz: Polyeder und Raumabteilungen, in: *Encyclopädie der Math. Wissenschaften* **3** (Geometrie) 3AB12, 1–139, 1922.

[216] K. Stephenson: *Introduction to Circle Packing, The Theory of Discrete Analytic Functions*, Cambridge University Press (2005).

[217] L. Surányi: On line-critical graphs, *Infinite and Finite Sets* III, (Colloq. Keszthely, Hungary, 1973), eds. A. Hajnal, R. Rado and V. T. Sós, Colloq. Math. Soc. János Bolyai **10**, North-Holland, Amsterdam (1975), 1411–1444.

[218] J.W. Sutherland: *Jahresbericht der Deutsch. Math. Vereinigung* **45** (1935), 2 Abt. 33.

[219] S. Straszewicz: Sur un problème geometrique de Erdős, *Bull. Acad. Polon. Sci. CI.III* **5** (1957), 39–40.

[220] M. Szegedy: A note on the θ number of Lovász and the generalized Delsarte bound, *Proc. 35th FOCS* (1994), 36–39.

[221] L.A. Székely: Measurable chromatic number of geometric graphs and sets without some distances in Euclidean space, *Combinatorica* **4** (1984), 213–218.

[222] L.A. Székely: Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* **6** (1997), 353–358.

[223] W.P. Thurston: *Three-dimensional Geometry and Topology*, Princeton Mathematical Series **35**, Princeton University Press, Princeton, NJ, 1997.

[224] W.P. Thurston: *The finite Riemann Mapping Theorem,* lecture at the International Symposium at Purdue University on the occasion of the proof of the Bieberbach Conjecture (1985).

[225] G. Tóth: Point sets with many k-sets, *Discr. Comput. Geom.* **26** (2001), 187–194.

[226] W.T. Tutte: How to draw a graph, *Proc. London Math. Soc.* **13** (1963), 743–768.

[227] V. Vassilevska Williams: Multiplying matrices faster than Coppersmith-Winograd,

[228] K. Vesztergombi: On the distribution of distances in finite sets in the plane, *Discrete Math.*, 57, [1985], 129–145.

[229] K. Vesztergombi: Bounds on the number of small distances in a finite planar set, *Stu. Sci. Acad. Hung.*, 22 [1987], 95–101.

[230] K. Vesztergombi: On the two largest distances in finite planar sets, *Discrete Math.* **150** (1996) 379–386.

[231] L. Vandeberghe and S. Boyd: Semidefinite programming, in: *Math. Programming: State of the Art* (ed. J. R. Birge and K. G. Murty), Univ. of Michigan, 1994.

[232] L. Vandeberghe and S. Boyd: Semidefinite programming. SIAM Rev. 38 (1996), no. 1, 49–95.

[233] R.J. Vanderbei and B. Yang: The simplest semidefinite programs are trivial, *Math. of Oper. Res.* **20** (1995), no. 3, 590–596.

[234] E. Welzl: Entering and leaving j-facets, *Discr. Comput. Geom.* **25** (2001), 351–364.

[235] W. Whiteley, Infinitesimally rigid polyhedra, *Trans. Amer. Math. Soc.* **285** (1984), 431–465.

[236] W. Whiteley: ***

[237] H. Wolkowitz: Some applications of optimization in matrix theory, *Linear Algebra and its Applications* **40** (1981), 101–118.

[238] H. Wolkowicz: Explicit solutions for interval semidefinite linear programs, *Linear Algebra Appl.* **236** (1996), 95–104.

[239] H. Wolkowicz, R. Saigal and L. Vandenberghe: *Handbook of semidefinite programming. Theory, algorithms, and applications.* Int. Ser. Oper. Res. & Man. Sci., **27** (2000) Kluwer Academic Publishers, Boston, MA.

[240] D. Zeilberger: A new basis for discrete analytic polynomials, *J. Austral. Math. Soc.* Ser. A **23** (1977), 95–104.

[241] D. Zeilberger: A new approach to the theory of discrete analytic functions, *J. Math. Anal. Appl.* **57** (1977), 350–367.

[242] D. Zeilberger and H. Dym: Further properties of discrete analytic functions, *J. Math. Anal. Appl.* **58** (1977), 405–418.

[243] R. Zippel: Probabilistic algorithms for sparse polynomials, in: *Symbolic and Algebraic Computation*, Lecture Notes in Computer Science **72** (1979), 216–226.