Chapter 3

Coin representation

3.1 Koebe's theorem

We prove Koebe's important theorem on representing a planar graph by touching circles [5], and its extension to a Steinitz representation, the Cage Theorem.

Theorem 3.1.1 (Koebe's Theorem) *Let G be a 3-connected planar graph. Then one can assign to each node i a circle C_i in the plane so that their interiors are disjoint, and two nodes are adjacent if and only if the corresponding circles are tangent.*

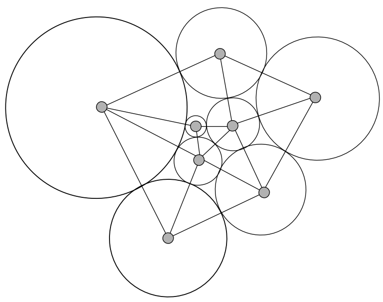

Figure 3.1: The coin representation of a planar graph

If we represent each of these circles by their center, and connect two of these centers by a segment if the corresponding circles touch, we get a planar map, which we call the *tangency graph* of the family of circles. Koebe's Theorem says that every planar graph is the tangency graph of a family of openly disjoint circular discs.

Koebe's Theorem was rediscovered and generalized by Andre'ev [1, 2] and Thurston [13]. One of these strengthens Koebe's Theorem in terms of a simultaneous representation of a 3-connected planar graph and of its dual by touching circles. To be precise, we define a *double* *circle representation in the plane* of a planar map G as two families of circles, $(C_i: i \in V)$ and $(D_p: p \in V^*)$ in the plane, so that for every edge ij, bordering countries p and q, the following holds: the circles C_i and C_j are tangent at a point $\mathbf{x}_{ij}$; the circles D_p and D_q are tangent at the same point $\mathbf{x}_{ij}$; and the circles D_p and D_q intersect the circles C_i and C_j at this point orthogonally. Furthermore, the interiors of the circular discs $\widehat{C}_i$ bounded by the circles C_i are disjoint and so are the disks $\widehat{D}_j$, except that the circle D_{p_0} representing the outer country contains all the other circles D_p in its interior.

Such a double circle representation has other useful properties.

Proposition 3.1.2 (a) *For every bounded country p, the circles D_p and C_i $(i \in V(p))$ cover p.* (b) *If $i \in V$ is not incident with $p \in V^*$, then $\widehat{C}_i$ and $\widehat{D}_p$ are disjoint.*

The proof of these facts is left to the reader as an exercise.

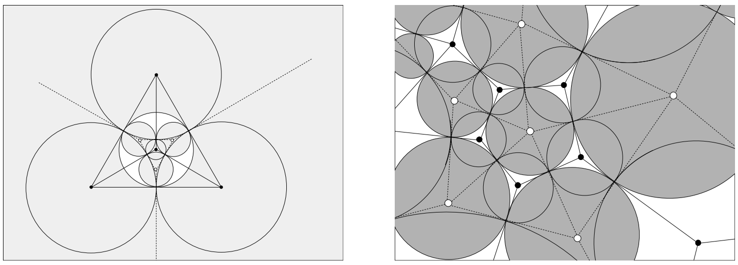

Figure 3.2: Two sets of circles, representing (a) K_4 and its dual (which is another K_4); (b) a planar graph and its dual.

Figure 3.2(a) shows this double circle representation of the simplest 3-connected planar graph, namely K_4. For one of the circles, the exterior domain should be considered as the disk it bounds. The other picture shows part of the double circle representation of a larger planar graph.

The main theorem is this chapter is that such representations exist.

Theorem 3.1.3 *Every 3-connected planar map G has a double circle representation in the plane.*

The proof is contained in the next sections.

3.1.1 Conditions on the radii

We fix a triangular country p_0 in G or G^* (say, G) as the outer face; let a, b, c be the nodes of p_0. For every node $i \in V$, let $F(i)$ denote the set of bounded countries containing i, and for every country p, let $V(p)$ denote the set of nodes on the boundary of p. Let $U = V \cup V^* \setminus \{p_0\}$, and let J denote the set of pairs ip with $p \in V^* \setminus \{p_0\}$ and $i \in V(p)$.

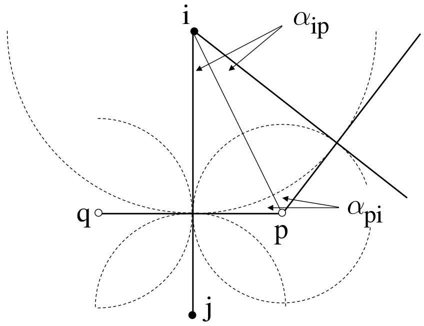

Figure 3.3: Notation.

Let us start with assigning a positive real number r_u to every node $u \in U$. Think of this as a guess for the radius of the circle representing u (we don't guess the radius for the circle representing p_0; this will be easy to add at the end). For every $ip \in J$, we define

$$\alpha_{ip} = \arctan\frac{r_p}{r_i} \qquad \text{and} \qquad \alpha_{pi} = \arctan\frac{r_i}{r_p} = \frac{\pi}{2} - \alpha_{ip}. \tag{3.1}$$

Suppose that i is an internal node. If the radii correspond to a correct double circle representation, then $2\alpha_{ip}$ is the angle between the two edges of the country p at i (Figure 3.3). Since these angels fill out the full angle around i, we have

$$\sum_{V(p)\ni i} \arctan\frac{r_p}{r_i} = \pi \qquad (i \in V \setminus \{a,b,c\}). \tag{3.2}$$

We can derive a similar conditions for the external nodes and the countries:

$$\sum_{V(p)\ni i} \arctan\frac{r_p}{r_i} = \frac{\pi}{6} \qquad (i \in \{a,b,c\}), \tag{3.3}$$

and

$$\sum_{i\in V(p)} \arctan\frac{r_i}{r_p} = \pi \qquad (p \in V^* \setminus \{p_0\}). \tag{3.4}$$

The key to the construction of a double circle representation is that these conditions are sufficient.

Lemma 3.1.4 *Suppose that the radii $r_u > 0$ $(u \in U)$ are chosen so that* (3.2), (3.3) *and* (3.4) *are satisfied. Then there is a double circle representation with these radii.*

Proof. Let us construct two right triangles with sides r_i and r_p for every $ip \in J$, one with each orientation, and glue these two triangles together along their hypotenuse to get a *kite* K_{ip}. Starting from a node i_1 of p_0, put down all kites K_{i_1p} in the order of the corresponding countries in a planar embedding of G. By (3.3), these will fill an angle of $\pi/3$ at i_1. Now proceed to a bounded country p_1 incident with i_1, and put down all the remaining kites $K_{p_1 i}$ in the order in which the nodes of p_1 follow each other on the boundary of p_1. By (3.4), these triangles will cover a neighborhood of p_1. We proceed similarly to the other bounded countries containing i_1, then to the other nodes of p_1, etc. The conditions (3.2), (3.3) and (3.4) will guarantee that we tile a regular triangle.

Let C_i be the circle with radius r_i about the position of node i constructed above, and define D_p analogously. We still need to define D_{p_0}. It is clear that p_0 is drawn as a regular triangle, and hence we necessarily have $r_a = r_b = r_c$. We define D_{p_0} as the inscribed circle of the regular triangle *abc*.

It is clear from the construction that we get a double circle representation of G. $\square$

Of course, we cannot expect conditions (3.2), (3.3) and (3.4) to hold for an arbitrary choice of the radii r_u. In the next sections we will see three methods to construct radii satisfying the conditions in Lemma 3.1.4; but first we give two simple lemmas proving something like that—but not quite what we want.

For a given assignment of radii r_u $(u \in U)$, consider the defects of the conditions in Lemma 3.1.4. To be precise, for a node $i \neq a, b, c$, define the defect by

$$\delta_i = \sum_{p \in F(i)} \alpha_{ip} - \pi.$$

For a boundary node $i \in \{a, b, c\}$, we modify this definition:

$$\delta_i = \sum_{p \in F(i)} \alpha_{ip} - \frac{\pi}{6}.$$

If p is a bounded face, we define its defect by

$$\delta_p = \sum_{i \in V(p)} \alpha_{pi} - \pi.$$

(Note that these defects may be positive or negative.) While of course an arbitrary choice of the radii r_u will not guarantee that all these defects are 0, the following lemma shows that this is true at least “one the average”:

Lemma 3.1.5 *For every assignment of radii, we have*

$$\sum_{u \in U} \delta_u = 0.$$

Proof. From the definition,

$$\sum_{u \in U} \delta_u = \sum_{i \in V \setminus \{a,b,c\}} \left(\sum_{p \in F(i)} \alpha_{ip} - \pi \right) + \sum_{i \in \{a,b,c\}} \left(\sum_{p \in F(i)} \alpha_{ip} - \frac{\pi}{6} \right)$$
$$+ \sum_{p \in V^* \setminus \{p_0\}} \left(\sum_{i \in V(p)} \alpha_{pi} - \pi \right).$$

Every pair $ip \in J$ contributes $\alpha_{ip} + \alpha_{pi} = \pi/2$. Since $|J| = 2m-3$, we get

$$(2m-3)\frac{\pi}{2} - (n-3)\pi - 3\frac{\pi}{6} - (f-1)\pi = (m-n-f+2)\pi.$$

By Euler's formula, this proves the lemma. □

Our second preliminary lemma shows that conditions (3.2), (3.3) and (3.4), considered as linear equations for the α_{ip}, can be satisfied.

Lemma 3.1.6 *Let G be a planar map with a triangular unbounded face $p_0 = abc$. Then there are real numbers $0 < \beta_{ip} < \pi/2$ ($p \in V^* \setminus \{p_0\}$, $i \in V(p)$) such that*

$$\sum_{V(p)\ni i} \beta_{ip} = \pi \qquad (i \in V \setminus \{a,b,c\}, \tag{3.5}$$

$$\sum_{V(p)\ni i} \beta_{ip} = \frac{\pi}{6} \qquad (i \in \{a,b,c\}, \tag{3.6}$$

and

$$\sum_{i\in V(p)} (\frac{\pi}{2} - \beta_{ip}) = \pi \qquad (p \in V^* \setminus \{p_0\}. \tag{3.7}$$

We don't claim here that the solutions are obtained in the form (3.1)!

Proof. Consider any straight line embedding of the graph (say, the Tutte rubber band embedding), with p_0 nailed to a regular triangle. For $i \in V(p)$, let β_{pi} denote the angle of the polygon p at the vertex i. Then the conclusions are easily checked. □

3.1.2 Reducing a defect function

In order to use the previous lemmas to prove Theorem 3.1.3, we have to construct find radii for the circles so that the defects are 0. There are several ways to do so. The proof we describe first is due to Colin de Verdière [4]. This is perhaps the shortest known proof, but it starts with a rather "ad hoc" step. One advantage is that we can use an "off the shelf" optimization algorithm for smooth convex functions to compute the representation. More combinatorial but lengthier proofs will be described in the next two sections.

Proof. We are going to look for the radii in the form $r_u = e^{x_u}$; this will guarantee that they are positive, and it also motivates at least the elements of the following definition:

$$\phi(x) := \int_{-\infty}^{x} \arctan(e^t)\, dt.$$

It is easy to verify that ϕ is monotone increasing, convex, and

$$\phi(x) = \max\{0, \frac{\pi}{2}x\} + O(1). \tag{3.8}$$

Let $x \in \mathbb{R}^V$, $y \in \mathbb{R}^{V^*}$. Using the numbers β_{ip} from Lemma 3.1.6, consider the function

$$F(x,y) = \sum_{i,p:\, p\in F(i)} \Big(\phi(y_p - x_i) - \beta_{ip}(y_p - x_i)\Big).$$

Claim 1 *If $|x|+|y| \to \infty$ while (say) $x_1 = 0$, then $F(x,y) \to \infty$.*

We need to fix one of the x_i, since if we add the came value to each x_i and y_p, then the value of F does not change. To prove the claim, we use (3.8):

$$\begin{aligned} F(x,y) &= \sum_{i,p:\, p\in F(i)} \Big(\phi(y_p - x_i) - \beta_{ip}(y_p - x_i)\Big) \\ &= \sum_{i,p:\, p\in F(i)} \Big(\max\{0, \frac{\pi}{2}(y_p - x_i)\} - \beta_{ip}(y_p - x_i)\Big) + O(1) \\ &= \sum_{i,p:\, p\in F(i)} \Big(\max\{-\beta_{ip}(y_p - x_i), (\frac{\pi}{2} - \beta_{ip})(y_p - x_i)\}\Big) + O(1). \end{aligned}$$

Since $-\beta_{ip}$ is negative but $\frac{\pi}{2} - \beta_{ip}$ is positive, each term here is nonnegative, and a given term tends to infinity if $|x_i - y_p| \to \infty$. If x_1 remains 0 but $|x|+|y| \to \infty$, then at least one difference $|x_i - y_p|$ must tend to infinity. This proves the Claim.

It follows from this Claim that F attains its minimum at some point (x,y), and here

$$\frac{\partial}{\partial x_i}F(x,y) = \frac{\partial}{\partial y_p}F(x,y) = 0 \qquad (i \in V, p \in V^* \setminus \{p_0\}). \tag{3.9}$$

Claim 2 *The radii*

$$r_i = e^{x_i} \quad (i \in V) \qquad \textit{and} \qquad r_p = e^{y_p} \quad (p \in V^*)$$

satisfy the conditions of Lemma 3.1.4.

Let i be an internal node, then

$$\frac{\partial}{\partial x_i}F(x,y) = -\sum_{p\in F(i)} \phi'(y_p - x_i) + \sum_{p\in F(i)} \beta_{ip} = -\sum_{p\in F(i)} \arctan(e^{y_p - x_i}) + \pi,$$

and so by (3.9),

$$\sum_{p\in F(i)} \arctan\frac{r_p}{r_i} = \sum_{p\in F(i)} \arctan(e^{y_p - x_i}) = \pi. \tag{3.10}$$

This proves (3.2). Conditions (3.3) and (3.4) follow by a similar computation.

Applying Lemma 3.1.4 completes the proof. □

3.1.3 The range of defects

The second proof is non-algorithmic, but it motivates the third, which is in a sense the most natural. We define a mapping $\Phi:\ (0,\infty)^U \to \mathbb{R}^U$ by

$$\Phi:\ (r_u:\ u\in U)\mapsto(\delta_u:\ u\in U). \tag{3.11}$$

The defect vector $(\delta_u:\ u\in U)$ depends on the ratios of the values r_u only; hence we may restrict our attention to positive radii satisfying $\sum_{u\in U} r_u=1$. Then the domain of the map δ is the interior of the $(n+f-2)$-dimensional simplex Σ defined by

$$x_u\ge 0,\qquad \sum_{u\in U} x_u=1.$$

Our next goal is to determine the range. By Lemma 3.1.4, all we need to show is that this range contains the origin.

We already know by Lemma 3.1.5 that the range of the mapping δ lies in the hyperplane defined by $\sum_u \delta_u=0$. We can derive some inequalities too. For $S\subseteq U$, define $J[S]=\{ip\in J:\ i,p\in S\}$ and

$$f(S)=\frac{1}{2}|J[S]|-|S|+\frac{5}{6}|S\cap\{a,b,c\}|.$$

Lemma 3.1.7 (a) $f(\emptyset)=f(U)=0$; (b) $f(S)<0$ *for every set* $\emptyset\subset S\subset U$.

Proof. The proof of (a) is left to the reader as Exercise 3.4.2.

To prove (b), consider the numbers β_{pi} in Lemma 3.1.6, and let $\beta_{ip}=\pi-\beta_{pi}$. We can do the following computation:

$$\begin{aligned}\pi|J(S)| &= \sum_{\substack{ip\in J\\ i,p\in S}}(\beta_{pi}+\beta_{ip})\\ &< \sum_{i\in S\cap V^*}\sum_{p\in V(i)}\beta_{ip}+\sum_{p\in S\cap V}\sum_{i\in F(p)}\beta_{pi}\\ &= 2\pi|S|-\frac{5}{3}\pi|S\cap\{a,b,c\}|.\end{aligned}$$

(The strict inequality comes from the fact that there is at least one pair ip where exactly one element belongs to S, and here we omitted a positive term β_{ip} or β_{pi}.) This proves the Lemma. □

Lemma 3.1.8 *For every set* $\emptyset\subset S\subset U$,

$$\sum_{u\in S}\delta_u>f(S)\pi.$$

Proof. We have

$$\begin{aligned}
\sum_{u\in S}\delta_u &= \sum_{p\in S\cap V}\sum_{i\in F(i)}\alpha_{pi} + \sum_{i\in S\cap V^*}\sum_{p\in V(i)}\alpha_{ip} \\
&\quad -|I(S)|\pi - |S\cap(V\setminus\{a,b,c\}|\pi - |S\cap\{a,b,c\}|\frac{\pi}{6} \\
&= \sum_{\substack{ip\in J\\ i\in S, p\notin S}}\alpha_{ip} + \sum_{\substack{ip\in J\\ i\notin S, p\in S}}\alpha_{pi} + \sum_{ip\in J[S]}(\alpha_{ip}+\alpha_{pi}) \qquad (3.12)\\
&\quad -|I(S)|\pi - |S\cap(V\setminus\{a,b,c\}|\pi - |S\cap\{a,b,c\}|\frac{\pi}{6} \\
&= \sum_{\substack{ip\in J\\ i\in S, p\notin S}}\alpha_{ip} + \sum_{\substack{ip\in J\\ i\notin S, p\in S}}\alpha_{pi} + f(S)\pi.
\end{aligned}$$

Since the first two terms are nonnegative, and (as in the previous proof) at least one is positive, the lemma follows. □

Now we are prepared to describe the range of defects. Let P denote the polyhedron in $\mathbb{R}^U$ defined by

$$\sum_{u\in U} x_u = 0, \qquad (3.13)$$

$$\sum_{u\in S} x_u \ge d(S)\pi \quad (\emptyset\subset S\subset U). \qquad (3.14)$$

Clearly P is bounded.

Lemma 3.1.9 *The origin is in the relative interior of P. The range of Φ is contained in P.*

Proof. The first statement follows by Lemma 3.1.8. For a vector in the range of Φ, equation (3.13) follows by Lemma 3.1.5, while inequality (3.14) follows by Lemma 3.1.7. □

Lemma 3.1.10 *The map Φ is injective.*

Proof. Consider two different choices r and r' of radii. Let S be the set of elements of U for which r'_u/r_u is maximum, then S is a nonempty proper subset of U. There is a pair $ip\in J$ ($i\in V^*$, $p\in V$) such that exactly one of i and p is in S; say, for example, that $i\in S$ and $p\notin S$. Then for every pair iq, $q\in V(i)$, we have $r'_q/r'_i\le r_q/r_i$, and strict inequality holds if $q=p$. Thus

$$\sum_{q\in V(i)}\arctan\frac{r'_q}{r'_i} < \sum_{q\in V(i)}\arctan\frac{r_q}{r_i},$$

showing that $\delta_i(r') < \delta_i(r)$, which proves the lemma. □

The next (technical) lemma is needed whenever we construct some assignment of radii as a limit of other such assignments.

Lemma 3.1.11 *Let $r^{(1)}, r^{(2)}, \dots \in \mathbb{R}^U$ be a sequence of assignments of radii, and let $\delta^{(k)} = \Phi(r^{(k)})$. Suppose that for each $u \in U$, $r_u^{(k)} \to \rho_u$ as $k \to \infty$. Let $S = \{u : \ \rho_u > 0\}$. Then*

$$\sum_{u \in S} \delta_u^{(k)} \to d(S)\pi \qquad (k \to \infty).$$

(We note that the right hand side is negative by Lemma 3.1.7.)

Proof. Recall the computation (3.12). The terms that result in strict inequality are α_{pi} with $p \in S$, $i \notin S$, and α_{ip} with $i \in S$, $p \notin S$. These terms tend to 0, so the slack in (3.13) tends to 0. □

Now we are able to prove the main result in this section.

Theorem 3.1.12 *The range of Φ is exactly the interior of P.*

Proof. Suppose not, and let y_1 be an interior point in P not in the range of Δ. Let y_2 be any point in the range. The segment connecting y_1 and y_2 contains a point y which is an interior point of P and on the boundary of the range of Δ. Consider a sequence $(r^1, r^2, \dots)$ with $\Delta(r^k) \to y$. We may assume (by compactness) that (r^k) is convergent. If (r^k) tends to an interior point r of Σ, then the fact that Δ is continuous, injective, and $\dim(P) = \dim(\Sigma)$ imply that the image of a neighborhood of r covers a neighborhood of y, a contradiction. If (r^k) tends to a boundary point of Σ, then by Lemma 3.1.11, $\delta(r_k)$ tends to the boundary of Σ, a contradiction. □

Since the origin is in the interior of P by Corollary 3.1.9, this implies that the origin is in the range of Φ. This completes the proof of Theorem 3.1.3.

3.1.4 Reducing the defect

The proof we saw is in the previous section an existence proof; it does not give an algorithm to construct the circles. In this section we describe an algorithmic proof. We'll need most of the lemmas proved above in the analysis of the algorithm.

We measure the "badness" of the assignment of the radii by the error

$$\mathcal{E} = \sum_{u \in U} \delta_u^2.$$

We want to modify the radii so that we reduce the error. The key observation is the following. Let i be a country with $\delta_i > 0$. Suppose that we increase the radius r_i, while keep the other radii fixed. Then α_{ip} decreases for every $p \in V(i)$, and correspondingly α_{pi} increases, but nothing else changes. Hence δ_i decreases, δ_p increases for every $p \in V(i)$, and all the other defects remain unchanged. Since the total defect remains the same by Lemma 3.1.5, we can describe this as follows: if we increase a radius, some of the defect of that node is distributed to its neighbors (in the $G^\diamondsuit$ graph). Note, however, that it is more difficult to describe in what proportion this defect is distributed (and we'll try to avoid to have to describe this).

How much of the defect of i can be distributed this way? If $r_i \to \infty$, then $\alpha_{ip} \to 0$ for every $p \in V(i)$, and so $\delta_i \to -\pi$. This means that we "overshoot". So we can distribute at least the positive part of the defect this way.

The same argument applies to the defects of nodes, and we can distribute negative defects similarly by decreasing the appropriate radius. Let us immediately renormalize the radii, so that we maintain that $\sum_u r_u = 1$. Recall that this does not change the defects.

There are many schemes that can be based on this observation: we can try to distribute the largest (positive) defect, or a positive defect adjacent to a node with negative defect etc. Brightwell and Scheinerman [3] prove that if we repeatedly pick any element $u \in U$ with positive defect and distribute all its defect, then the process will converge to an assignment of radii with no defect. There is a technical hurdle to overcome: since the process is infinite, one must argue that no radius tends to 0 or ∞. We give a somewhat different argument.

Consider a subset $\emptyset \subset S \subset U$, and multiply each radius r_u, $u \notin S$, by the same factor $0 < \lambda < 1$. Then α_{ip} is unchanged if both i and p are in S or outside S; if $i \in S$ and $p \notin S$ then α_{ip} decreases while α_{pi} increases; and similarly the other way around. Hence δ_u does not increase if $u \in S$ and strictly decreases if $u \in S$ has a neighbor outside S. Similarly, δ_u does not decrease if $u \in U \setminus S$ and strictly increases if $u \in U \setminus S$ has a neighbor in S.

Let $\emptyset \subset S \subset U$ be a set such that $\min_{u\in S} \delta_u > \max_{U\in U\setminus S} \delta_u$. We claim that we can decrease the radii in $U \setminus S$ until one of the δ_u, $u \in S$, becomes equal to a δ_v, $v \notin S$. If not, then (after renormalization) the radii in $U \setminus S$ would tend to 0 while still any defect in S would be larger than any defect in $U \setminus S$. But it follows by Lemmas 3.1.7 and 3.1.11) that in this case the total defect in S tends to a negative value, and so the total defect in $U \setminus S$ tends to a positive value. So there is an element of S with negative defect and an element of $U \setminus S$ with positive defect, which is a contradiction.

Let t be this common value, and let δ'_u be the new defects. Then the change in the error is

$$\mathcal{E} - \mathcal{E}' = \sum_{u\in U} \delta_u^2 - \sum_{u\in U} {\delta'_u}^2$$

Using Lemma 3.1.5, we can write this in the form

$$\mathcal{E} - \mathcal{E}' = \sum_{u\in U} (\delta_u - \delta'_u)^2 + 2 \sum_{u\in U} (t - \delta'_u)(\delta'_u - \delta_u).$$

By the choice of t, we have $t \le \delta'_u$ and $\delta_u \ge \delta'_u$ for $u \in S$ and $t \ge \delta'_u$ and $\delta_u \le \delta'_u$ for $u \notin S$. Hence the second sum in $\mathcal{E} - \mathcal{E}'$ is nonnegative, while the first is positive. So the error decreases; in fact, it decreases by at least $(\delta_u - t)^2 + (\delta_v - t)^2 \ge (\delta_u - \delta_v)^2/4$ for some $u \in S$ and $v \in U \setminus S$. If we choose the largest gap in the sequence of the δ_u ordered decreasingly, then this gain is at least

$$\left(\frac{1}{m}(\max\delta_u - \min\delta_u)\right)^2 \geq \left(\frac{1}{m}\sqrt{\frac{\mathcal{E}}{m}}\right)^2 = \frac{\mathcal{E}}{m^3}.$$

Thus we have

$$\mathcal{E}(r') \leq \left(1 - \frac{1}{m^3}\right)\mathcal{E}(r). \tag{3.15}$$

If we iterate this procedure, we get a sequence of vectors $r^1, r^2, \dots \in \Sigma$ for which $\mathcal{E}(r^k) \to 0$. No subsequence of the r^k can tend to a boundary point of Σ. Indeed, by Lemma 3.1.11, for such a sequence $\delta(r^{(k)})$ would tend to the boundary of P, and so by Corollary 3.1.9, the error would stay bounded away from 0. Similarly, if a subsequence tends to an interior point $r \in \Sigma$, then $\delta(r) = 0$, and by Claim 3.1.10, this limit is unique. It follows that there is a (unique) point r in the interior of Σ with $\delta(r) = 0$, and the sequence r^k tends to this point.

3.2 Formulation in space

3.2.1 The Cage Theorem

One of the nicest consequences of the coin representation (more exactly, the double circle representation in Theorem 3.1.3) is the following theorem, due to Andre'ev [1], which is a rather far reaching generalization of the Steinitz Representation Theorem.

Theorem 3.2.1 (The Cage Theorem) *Every 3-connected planar graph can be represented as the skeleton of a convex* 3*-polytope such that every edge of the polytope touches a given sphere.*

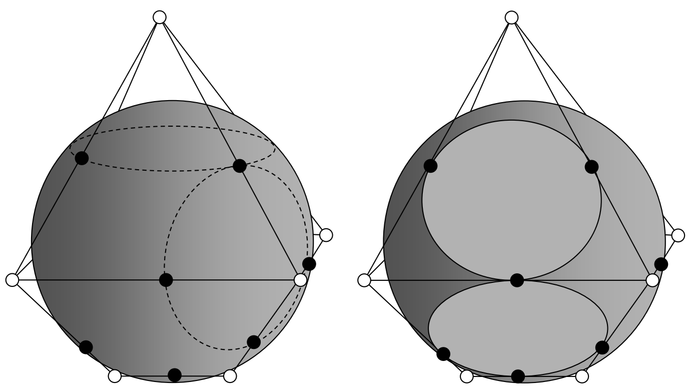

Figure 3.4: A convex polytope in which every edge is tangent to a sphere creates two families of circles on the sphere.

Proof. Let $G = (V, E)$ be a 3-connected planar map, and let p_0 be its unbounded face. Let $(C_i : i \in V)$ and $D_j : j \in V^*)$ be a double circle representation of G in the plane. Take a sphere S touching the plane at the center $\mathbf{c}$ of D_{q_0}, where $q_0 \in V^* \setminus \{p_0\}$. Consider the point of tangency as the south pole of S. Project the plane onto the sphere from the north pole (the antipode of $\mathbf{c}$). This transformation, called *inverse stereographic projection*, has many useful properties: it maps circles and lines onto circles, and it preserves the angle between them. It will be convenient to choose the radius of the sphere so that the image of D_{q_0} is contained in the southern hemisphere, but the image of the interior of D_{p_0} covers more than a hemisphere.

Let $(C'_i : i \in V)$ and $D'_j : j \in V^*)$ be the images of the circles in the double circle representation. We define caps $(\widehat{C}_i : i \in V)$ and $\widehat{D}_j : j \in V^*)$ with boundaries C_i and D_j, respectively: we assign the cap not containing the north pole to every circle except to D_{p_0}, to which we assign the cap containing the north pole. This way we get two families of caps on the sphere. Every cap covers less than a hemisphere, since the caps $\widehat{C}_i$ $(i \in V)$ and $\widehat{D}_p$ $(p \in V^* \setminus \{p_0, q_0\})$ miss both the north pole and the south pole, and $\widehat{D}_{p_0}$ and $\widehat{D}_{q_0}$ have this property by the choice of the radius of the sphere. The caps $\widehat{C}_i$ are openly disjoint, and so are the caps $\widehat{D}_j$. Furthermore, for every edge $ij \in E(G)$, the caps $\widehat{C_i}$ and $\widehat{C_j}$ are tangent to each other, and so are the caps $\widehat{D_p}$ and $\widehat{D_q}$ representing the endpoints of the dual edge pq, the two points of tangency are the same, and C'_i and C'_j are orthogonal to D'_p and D'_q. The tangency graph of the caps C_i, drawn on the sphere by arcs of large circles, is isomorphic to the graph G.

This nice picture translates into polyhedral geometry as follows. Let $\mathbf{u}_i$ be the point above the the center of $\widehat{C}_i$ whose "horizon" is the circle C'_i, and let $\mathbf{v}_p$ be defined analogously for $p \in V^*$. Let $ij \in E(G)$, and let pq be the corresponding edge of G^*. The points $\mathbf{u}_i$ and $\mathbf{u}_j$ are contained in the tangent of the sphere that is orthogonal to the circles C'_i and C'_j and their common point x; this is clearly the same as the common tangent of D'_p and D'_q at x. The plane $\mathbf{v}_p^\mathsf{T} x = 1$ intersects the sphere in the circle D'_p, and hence it contains it tangents, in particular the points $\mathbf{u}_i$ and $\mathbf{u}_j$, and similarly, all points $\mathbf{u}_k$ where k is a node of the facet p. Since the cap $\widehat{D}_p$ is disjoint from $\widehat{C}_k$ if k is not a node of the facet p, we have $\mathbf{v}_p^\mathsf{T}\mathbf{u}_k < 1$ for every such node.

This implies that the polytope $P = \text{conv}\{\mathbf{u}_i : i \in V\}$ is contained in the polyhedron $P' = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{v}_p^\mathsf{T}\mathbf{x} \le 1 \ \forall p \in V^*\}$. Furthermore, every inequality $\mathbf{v}_p^\mathsf{T}\mathbf{x} \le 1$ defines a facet F_p of P with vertices $\mathbf{u}_i$, where i is a node of p. Every ray from the origin intersects one of the countries p in the drawing of the graph on the sphere, and therefore it intersects F_p. This implies that P has no other facets, and thus $P = P'$. It also follows that every edge of P is connecting two vertices $\mathbf{u}_i$ and $\mathbf{u}_j$, where $ij \in E$, and hence the skeleton of P is isomorphic to G and every edge is tangent to the sphere. $\square$

Conversely, the double circle representation follows easily from this theorem, and the argument is obtained basically by reversing the proof above. Let P be a polytope as in the theorem. It will be convenient to assume that the center of the sphere is in the interior of P; this can be achieved by an appropriate projective transformation of the space (we come back to this in Section 3.2.2).

We start with constructing a double circle representation on the sphere: each node is represented by the horizon on the sphere when looking from the corresponding vertex, and each facet is represented by the intersection of its plane with the sphere (Figure 3.4). Elementary geometry tells us that the circles representing adjacent nodes touch each other at the point where the edge touches the sphere, and the two circles representing the adjacent countries also touch each other at this point, and they are orthogonal to the first two circles. Furthermore, the interiors (smaller sides) of the horizon-circles are disjoint, and the same holds for the facet-circles. These circles can be projected to the plane by stereographic projection. It is easy to check that the projected circles form a double circle representation.

Remark 3.2.2 There is another polyhedral representation that can be read off from the double circle representation. Let $(C_i\,:\;\; i \in V)$ and $D_p\,:\;\; p \in V^*)$ form a double circle representation of $G = (V, E)$ on the sphere S, and for simplicity assume that interiors (smaller sides) of the C_i are disjoint, and the same holds for the interiors of the D_p. We can represent each circle C_i as the intersection of a sphere A_i with S, where A_i is orthogonal to S. Similarly, we can write $D_p = B_p \cap S$, where B_p is a sphere orthogonal to S. Let $\widehat{A}_i$ and $\widehat{B}_p$ denote the corresponding closed balls.

Let P denote the set of points in the interior of S that are not contained in any $\widehat{A}_i$ and $\widehat{B}_p$. This set is open and nonempty (since it contains the origin). The balls $\widehat{A}_i$ and $\widehat{B}_p$ cover the sphere S, and even their interiors do so with the exception of the points $\mathbf{x}_{ij}$ where two circles C_i and C_j touch. It follows that P is a domain whose closure $\overline{P}$ contains a finite number of points of the sphere S. It also follows that no three of the sphere A_i and B_p have a point in common except on the sphere S. Hence those points in the interior of S that belong to two of these spheres form circular arcs that go from boundary to boundary, and it is easy to see that they are orthogonal to S. For every incident pair (i, p) $(i \in V, p \in V^*)$ there is such an "edge" of P. The "vertices" of P are the points $\mathbf{x}_{ij}$, which are all on the sphere S, and together with the "edges" as described above they form a graph isomorphic to the medial graph G^+ of G.

All this translates very nicely, if we view the interior of S as a Poincaré model of the 3-dimensional hyperbolic space. In this model, spheres orthogonal to S are "planes", and hence P is a polyhedron. All the "vertices" of P which are at infinity, but if we allow them, then P is a Steinitz representation of G_+ in hyperbolic space. Every dihedral angle of P is $\pi/2$.

Conversely, a representation of G_+ in hyperbolic space as a polyhedron with dihedral angles $\pi/2$ and with all vertices at infinity gives rise to a double circle representation of G (in the plane or on the sphere, as you wish). Andreev gave a general necessary and sufficient condition for the existence of representation of a planar graph by a polyhedron in hyperbolic 3-space with prescribed dihedral angles. From this representation, he was able to derive Theorems 3.2.1 and 3.1.3.

Schramm [10] proved the following very general extension of Theorem 3.2.1, for whose proof we refer to the paper.

Theorem 3.2.3 (Caging the Egg) *For every smooth strictly convex body K in $\mathbb{R}^3$, every 3-connected planar graph can be represented as the skeleton of a polytope in $\mathbb{R}^3$ such that all of its edges touch K.* □

3.2.2 Conformal transformations

The double circle representation of a planar graph is uniquely determined, once a triangular unbounded country is chosen and the circles representing the nodes of this triangular country are fixed. This follows from Lemma 3.1.10. Similar assertion is true for double circle representations in the sphere. However, in the sphere there is no difference between faces, and we may not want to "normalize" by fixing a face. Often it is more useful to apply a circle-preserving transformation that distributes the circles on the sphere in a "uniform" way. The following Lemma shows that this is possible with various notions of uniformity.

Lemma 3.2.4 *Let $F : (B^3)^n \to B^3$ be a continuous map with the property that whenever $n-2$ of the vectors $\mathbf{u}_i$ are equal to $\mathbf{v} \in S^2$ then $\mathbf{v}^\mathsf{T} F(\mathbf{u}_1, \dots, \mathbf{u}_n) \geq 0$. Let $(C_1, \dots, C_n)$ be a family of openly disjoint caps on the sphere. Then there is a circle preserving transformation τ of the sphere such that $F(\mathbf{v}_1, \dots, \mathbf{v}_n) = 0$, where $\mathbf{v}_i$ is the center of $\tau(C_i)$.*

Examples of functions F to which this lemma applies are the center of gravity of $\mathbf{u}_1, \dots, \mathbf{u}_n$, or the center of gravity of their convex hull, or the center of the inscribed ball of this convex hull.

Proof. For every interior point $\mathbf{x}$ of the unit ball, we define a conformal (circle-preserving) transformation $\tau_\mathbf{x}$ of the unit sphere such that if $\mathbf{x}_k \to \mathbf{p} \in S^2$, then $\tau_{\mathbf{x}_k}(\mathbf{y}) \to -\mathbf{p}$ for every $\mathbf{y} \in S^2$, $\mathbf{y} \neq \mathbf{p}$.

We can define such maps as follows. For $\mathbf{x} = 0$, we define $\tau_\mathbf{x} = \mathrm{id}_B$. If $\mathbf{x} \neq 0$, then we take a tangent plane T at $\mathbf{x}^0$, and project the sphere stereographically onto T; blow up the plane from center $\mathbf{x}^0$ by a factor of $1/(1-|\mathbf{x}|)$; and project it back stereographically to the sphere. Let $\mathbf{v}_i(\mathbf{x})$ denote the center of the cap $\tau_\mathbf{x}(C_i)$. (Warning: this is not the image of the center of C_i in general! Conformal transformations don't preserve the centers of circles.)

We want to show that the range of $F(\mathbf{v}_1(\mathbf{x}),\dots,\mathbf{v}_n(\mathbf{x}))$ (as a function of $\mathbf{x}$) contains the origin. Suppose not. For $0<t<1$ and $\mathbf{x}\in tB$, define

$$f_t(\mathbf{x}) = tF(\mathbf{v}_1(\mathbf{x}),\dots,\mathbf{v}_n(\mathbf{x}))^0.$$

Then f_t is a continuous map $tB\to tB$, and so by Brouwer's Fixed Point Theorem, it has a fixed point $\mathbf{p}_t$, satisfying

$$\mathbf{p}_t = tF(\mathbf{v}_1(\mathbf{p}_t),\dots,\mathbf{v}_n(\mathbf{p}_t))^0.$$

Clearly $|\mathbf{p}_t| = t$. We may select a sequence of numbers $t_k\in(0,1)$ such that $t_k\to 1$, $\mathbf{p}_{t_k}\to \mathbf{q}\in B$ and $\mathbf{v}_i(\mathbf{p}_t)\to\mathbf{w}_i\in B$ for every i. Clearly $|\mathbf{q}|=1$ and $|\mathbf{w}_i|=1$. By the continuity of F, we have $F(\mathbf{v}_1(\mathbf{p}_t),\dots,\mathbf{v}_n(\mathbf{p}_t)\to F(\mathbf{w}_1,\dots,\mathbf{w}_n)$, and hence $\mathbf{q}=F(\mathbf{w}_1,\dots,\mathbf{w}_n)^0$. From the properties of $\tau_{\mathbf{x}}$ it follows that if C_i does not contain $\mathbf{q}$, then $\mathbf{w}_i=-\mathbf{q}$. Since at most two of the discs C_i contain $\mathbf{q}$, at most two of $\{\mathbf{w}_1,\dots,\mathbf{w}_n\}$ are different from $-\mathbf{q}$, and hence $(-\mathbf{q})^{\mathsf{T}}F(\mathbf{w}_1,\dots,\mathbf{w}_n)^0 = -\mathbf{q}^{\mathsf{T}}\mathbf{q}\ge 0$ by our assumption about F. This is a contradiction. □

3.3 Applications of coin representations

3.3.1 Planar separators

Koebe's Theorem has several important applications. We start with a simple proof by Miller and Thurston [7] of the Planar Separator Theorem 3.3.1 of Lipton and Tarjan [6] (we present the proof with a weaker bound of $3n/4$ on the sizes of the components instead of $2n/3$; see [12] for an improved analysis of the method).

Theorem 3.3.1 (Planar Separator Theorem) *Every planar graph $G=(V,E)$ on n nodes contains a set $S\subseteq V$ such that $|S|\le 4\sqrt{n}$, and every connected component of $G\setminus S$ has at most $2n/3$ nodes.*

We need the notion of the "statistical center", which is important in many other studies in geometry. Before defining it, we prove a simple lemma.

Lemma 3.3.2 *For every set $S\subseteq\mathbb{R}^d$ of n points there is a point $c\in\mathbb{R}^n$ such that every closed halfspace containing c contains at least $n/(d+1)$ elements of S.*

Proof. Let $\mathcal{H}$ be the family of all closed halfspaces that contain more than $dn/(d+1)$ points of S. The intersection of any $(d+1)$ of these still contains an element of S, so in particular it is nonempty. Thus by Helly's Theorem, the intersection of all of them is nonempty. We claim that any $c\in\cap\mathcal{H}$ satisfies the conclusion of the Lemma.

If H be any open halfspace containing c, then $\mathbb{R}^d\setminus H\notin\mathcal{H}$, which means that H contains at least $n/(d+1)$ points of S. If H is a closed halfspace containing c, then it is contained in an open halfspace H' that intersects S in exactly the same set, and applying the previous argument to H' we are done. □

A point c as in Lemma 3.3.2 is sometimes called a "statistical center" of the set S. To make this point well-defined, we call the center of gravity of all points c satisfying the conclusion of Lemma 3.3.2 *the statistical center* of the set (note: points c satisfying the conclusion of Lemma form a convex set, whose center of gravity is well defined).

Proof of Theorem 3.3.1. Let $(C_i : i \in V)$ be a Koebe representation of G on the unit sphere, and let $\mathbf{u}_i$ be the center of C_i on the sphere, and ρ_i, the spherical radius of C_i. By Lemma 3.2.4, we may assume that the statistical center of the points $\mathbf{u}_i$ is the origin.

Take any plane H through 0. Let S denote the set of nodes i for which C_i intersects H, and let S_1 and S_2 denote the sets of nodes for which C_i lies on one side and the other of H. Clearly there is no edge between S_1 and S_2, and so the subgraphs G_1 and G_2 are disjoint and their union is $G \setminus S$. Since 0 is a statistical center of the $\mathbf{u}_i$, it follows that $|S_1|, |S_2| \le 3n/4$.

It remains to make sure that S is small. To this end, we choose H at random, and estimate the expected size of S.

What is the probability that H intersects C_i? If $\rho_i \ge \pi/2$, then this probability is 1, but there is at most one such node, so we can safely ignore it, and suppose that $\rho_i < \pi/2$ for every i. By symmetry, instead of fixing C_i and choosing H at random, we can fix H and choose the center of C_i at random. Think of H as the plane of the equator. Then C_i will intersect H if and only if it is center is at a latitude at most ρ_i (North or South). The area of this belt around the equator is, by elementary geometry, $4\pi \sin \rho_i$, and so the probability that the center of C_i falls into here is $2 \sin \rho_i$. It follows that the expected number of caps C_i intersected by H is $\sum_{i \in V} 2 \sin \rho_i$.

To get an upper bound on this quantity, we use the surface area of the cap C_i is $2\pi(1 - \cos \rho_i) = 4\pi \sin^2(\rho_i/2)$, and since these are disjoint, we have

$$\sum_{i \in V} \left(\sin \frac{\rho_i}{2} \right)^2 < 1. \tag{3.16}$$

Using that $\sin \rho_i \le 2 \sin \frac{\rho_i}{2}$, we get by Cauchy-Schwartz

$$\sum_{i \in V} 2 \sin \rho_i \le 2\sqrt{n} \left(\sum_{i \in V} (\sin \rho_i)^2 \right)^{1/2} \le 4\sqrt{n} \left(\sum_{i \in V} \left(\sin \frac{\rho_i}{2} \right)^2 \right)^{1/2} < 4\sqrt{n}.$$

So the expected size of S is less than $4\sqrt{n}$, and so there is at least one choice of H for which $|S| < 4\sqrt{n}$. □

3.3.2 Laplacians of planar graphs

The Planar Separator theorem was first proved by direct graph-theoretic arguments; but for the following theorem on the eigenvalue gap of the Laplacian of planar graphs by Spielman and Teng [11] there is no proof known avoiding Koebe's theorem.

Theorem 3.3.3 *For every connected planar graph $G = (V, E)$ on n nodes and maximum degree D, the second smallest eigenvalue of L_G is at most $8D/n$.*

Proof. Let $C_i : i \in V$ be a Koebe representation of G on the unit sphere, and let $\mathbf{u}_i$ be the center of C_i, and ρ_i, the spherical radius of C_i. By Lemma 3.2.4 may assume that $\sum_i \mathbf{u}_i = 0$.

The second smallest eigenvalue of L_G is given by

$$\lambda_2 = \min_{\substack{x \neq 0 \\ \sum_i x_i = 0}} \frac{\sum_{ij \in E} (x_i - x_j)^2}{\sum_{i \in V} x_i^2},$$

Let $\mathbf{u}_i = (u_{i1}, u_{i2}, u_{i3})$, then this implies that

$$\sum_{ij \in E} (u_{ik} - u_{jk})^2 \geq \lambda_2 \sum_{i \in V} u_{ik}^2$$

holds for every coordinate k, and summing over k, we get

$$\sum_{ij \in E} \|\mathbf{u}_i - \mathbf{u}_j\|^2 \geq \lambda_2 \sum_{i \in V} \|\mathbf{u}_i\|^2 = \lambda_2 n. \tag{3.17}$$

On the other hand, we have

$$\begin{aligned} \|\mathbf{u}_i - \mathbf{u}_j\|^2 &= 4 \left(\sin \frac{\rho_i + \rho_j}{2} \right)^2 = 4 \left(\sin \frac{\rho_i}{2} \cos \frac{\rho_j}{2} + \sin \frac{\rho_j}{2} \cos \frac{\rho_i}{2} \right)^2 \\ &\leq 4 \left(\sin \frac{\rho_i}{2} + \sin \frac{\rho_j}{2} \right)^2 \leq 8 \left(\sin \frac{\rho_i}{2} \right)^2 + 8 \left(\sin \frac{\rho_j}{2} \right)^2, \end{aligned}$$

and so by (3.16)

$$\sum_{ij \in E} \|\mathbf{u}_i - \mathbf{u}_j\|^2 \leq 8D \sum_{i \in V} \left(\sin \frac{\rho_i}{2} \right)^2 \leq 8D.$$

Comparison with (3.17) proves the theorem. □

This theorem says that planar graphs are very bad expanders. The result does not translate directly to eigenvalues of the adjacency matrix or the transition matrix of the random walk on G, but for graphs with bounded degree it does imply the following:

Corollary 3.3.4 *Let G be a connected planar graph on n nodes with maximum degree D. Then the second largest eigenvalue of the transition matrix is at least $1 - 8D/n$, and the mixing time of the random walk on G is at least $\Omega(n/D)$.*

3.4 Circle packing and the Riemann Mapping Theorem

Koebe's Circle Packing Theorem and the Riemann Mapping Theorem in complex analysis are closely related. More exactly, we consider the following generalization of the Riemann Mapping Theorem.

Theorem 3.4.1 (The Koebe-Poincaré Uniformization Theorem) *Every open domain in the sphere whose complement has a finite number of connected components is conformally equivalent to a domain obtained from the sphere by removing a finite number of disjoint disks and points.*

The Circle Packing Theorem and the Uniformization Theorem are mutually limiting cases of each other (Koebe [5], Rodin and Sullivan [8]). The exact proof of this fact has substantial technical difficulties, but it is not hard to describe the idea.

1. To see that the Uniformization Theorem implies the Circle Packing Theorem, let G be a planar map and G^* its dual. We may assume that G and G^* are 3-connected, and that G^* has straight edges (these assumptions are not essential, just convenient). Let $\varepsilon > 0$, and let U denote the ε-neighborhood of G^*. By Theorem 3.4.1, there is a conformal map of U onto a domain $D' \subseteq S^2$ which is obtained by removing a finite number of disjoint caps and points from the sphere (the removed points can considered as degenerate caps). If ε is small enough, then these caps are in one-to-one correspondence with the nodes of G. We normalize using Lemma 3.2.4 and assume that the center of gravity of the cap centers is 0.

Letting $\varepsilon \to 0$, we may assume that the cap representing any given node $v \in V(G)$ converges to a cap C_v. One can argue that these caps are non-degenerate, caps representing different nodes tend to openly disjoint caps, and caps representing adjacent nodes tend to caps that are touching.

2. In the other direction, let $U = S^2 \setminus K_1 \setminus \cdots \setminus K_n$, where $K_1, \dots, K_n$ are disjoint closed connected sets which don't separate the sphere. Let $\varepsilon > 0$. It is not hard to construct a family $\mathcal{C}(\varepsilon)$ of openly disjoint caps such that the radius of each cap is less than ε and their tangency graph G is a triangulation of the sphere.

Let H_i denote the subgraph of G consisting of those edges intersecting K_i'. If ε is small enough, then the subgraphs H_i are node-disjoint, and each H_i is nonempty except possibly if K_i is a singleton. It is also easy to see that the subgraphs H_i are connected.

Let us contract each nonempty connected H_i to a single node w_i. If K_i is a singleton set and H_i is empty, we add a new node w_i to G in the triangle containing K_i', and connect it to the nodes of this triangle. The spherical map G' obtained this way can be represented as the tangency graph of a family of caps $\mathcal{D} = \{D_u : u \in V(G')\}$. We can normalize so that the center of gravity of the centers of $D_{w_1}, \dots, D_{w_n}$ is the origin.

Now let $\varepsilon \to 0$. We may assume that each $D_{w_i} = D_{w_i}(\varepsilon)$ tends to a cap $D_{w_i}(0)$. Furthermore, we have a map f_ε that assigns to each node u of G_ε the center of the corresponding cap D_u. One can prove (but this is nontrivial) that these maps f_ε, in the limit as $\varepsilon \to 0$, give a conformal map of U onto $S^2 \setminus D_{w_1}(0) \setminus \cdots \setminus D_{w_n}(0)$.

Exercise 3.4.2 Prove Lemma 3.1.7.

Exercise 3.4.3 Prove that every double circle representation of a planar graph is a rubber band representation with appropriate rubber band strengths.

Exercise 3.4.4 Show by an example that the bound in Lemma 3.3.2 is sharp.

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