Chapter 1

Rubber bands

In the previous chapter, we already used the idea of looking at the graph geometrically, by placing its nodes on the line and replacing the edges by rubber bands. Since, however, the positions of the nodes could be described by real numbers, this was more a visual support for an algebraic treatment than geometry. In this chapter we study a similar construction in higher dimensions, where the geometric point of view will play a more important role.

A *geometric representation* in d dimensions of a graph $G = (V, E)$ (often also called a *vector labeling*) is a map $\mathbf{x} : V \to \mathbb{R}^d$. We will also write $(\mathbf{x}_i : i \in V)$ for such a representation. At this time, we don't assume that the mapping is injective; this will be a pleasant property to have, but not always achievable. Needless to say, the geometric representation is interesting only if the position of the nodes have something to do with the structure of the graph, and this is what we will explore.

A representation is *in general position* if any $d+1$ representing points are affine independent. Sometimes we need a stronger condition: we say that the representation is *generic*, if all coordinates of the representing points are algebraically independent real numbers.

1.1 Rubber band representation

Let $G = (V, E)$ be a connected graph and $\emptyset \neq S \subseteq V$. Fix an integer $d \geq 1$ and a map $\mathbf{x}^0 : S \to \mathbb{R}^d$. We extend this to a representation $\mathbf{x} : V \to \mathbb{R}^d$ (a geometric representation of G) as follows.

First, let's give an informal description. Replace the edges by ideal rubber bands (satisfying Hooke's Law). Think of the nodes in S as nailed to their given position (node $i \in S$ to $\mathbf{x}_i^0 \in \mathbb{R}^d$), but let the other nodes settle in equilibrium. (We are going to see that this equilibrium position is uniquely determined.) We call this equilibrium position of the nodes the *rubber band representation* of G in $\mathbb{R}^d$ extending $\mathbf{x}^0$. The nodes in S will be called *nailed*, and the other nodes, *free* (Figure 1.1).

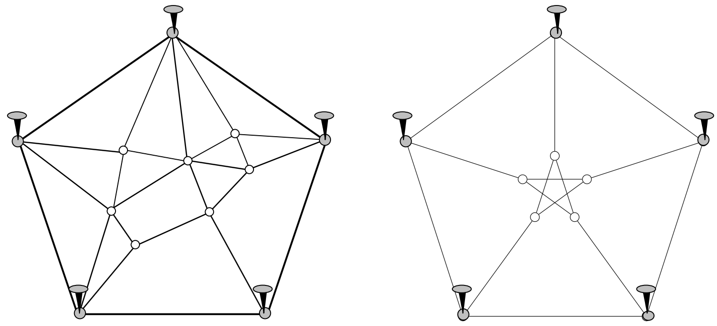

Figure 1.1: Rubber band representation of a planar graph and of the Petersen graph.

To be precise, let $\mathbf{x}_i = (x_{i1}, \dots, x_{id})^\mathsf{T} \in \mathbb{R}^d$ be the position of node $i \in V$. By definition, $\mathbf{x}_i = \mathbf{x}_i^0$ for $i \in S$. The *energy* of this representation is defined as

$$\mathcal{E}(\mathbf{x}) = \sum_{ij \in E} |\mathbf{x}_i - \mathbf{x}_j|^2 = \sum_{ij \in E} \sum_{k=1}^{d} (x_{ik} - x_{jk})^2. \tag{1.1}$$

We want to find the representation with minimum energy, subject to the boundary conditions:

$$\text{minimize } \mathcal{E}(\mathbf{x}) \tag{1.2}$$

$$\text{subject to } \mathbf{x}_i = \mathbf{x}_i^0 \text{ for all } i \in S. \tag{1.3}$$

Lemma 1.1.1 *The function $\mathcal{E}(\mathbf{x})$ is strictly convex.*

Proof. In (1.1), every function $(x_{ik} - x_{jk})^2$ is convex, so $\mathcal{E}$ is convex. Suppose that it is not strictly convex, which means that $\mathcal{E}(\frac{\mathbf{x}+\mathbf{y}}{2}) = \frac{1}{2}(\mathcal{E}(\mathbf{x}) + \mathcal{E}(\mathbf{y}))$ for two representations $\mathbf{x} \neq \mathbf{y} : V \to \mathbb{R}^d$. Then for every edge ij and every $1 \le k \le d$ we have

$$\left(\frac{x_{ik} + y_{ik}}{2} - \frac{x_{jk} + y_{jk}}{2}\right)^2 = \frac{(x_{ik} - x_{jk})^2 + (x_{ik} - x_{jk})^2}{2},$$

which implies that $x_{ik} - y_{ik} = x_{jk} - y_{jk}$. Since $x_{ik} = y_{ik}$ for every $i \in S$, using that G is connected and S is nonempty it follows that $x_{ik} = y_{ik}$ for every $i \in V$. So $\mathbf{x} = \mathbf{y}$, which is a contradiction. □

It is trivial that if any of the $\mathbf{x}_i$ tends to infinity, then $\mathcal{E}(\mathbf{x})$ tends to infinity (still assuming the boundary conditions 1.3 hold, where S is nonempty). With Lemma 1.1.1 this implies that the representation with minimum energy is uniquely determined. If $i \in V \setminus S$, then at the representation with minimal energy the partial derivative of $\mathcal{E}(\mathbf{x})$ with respect to any coordinate of $\mathbf{x}$ must be 0. This means that for every $i \in V \setminus S$,

$$\sum_{j \in N(i)} (\mathbf{x}_i - \mathbf{x}_j) = 0. \tag{1.4}$$

We can rewrite this as

$$\mathbf{x}_i = \frac{1}{d_i} \sum_{j \in N(i)} \mathbf{x}_j. \tag{1.5}$$

This equation means that *every free node is in the center of gravity of its neighbors.* Equation (1.4) also has a nice physical meaning: the rubber band connecting i and j pulls i with force $\mathbf{x}_j - \mathbf{x}_i$, so (1.4) states that the forces acting on i sum to 0 (as they should at the equilibrium).

It will be useful to extend the rubber band construction to the case when the edges of G have arbitrary positive weights (or "strengths"). Let $w_{ij} > 0$ denote the strength of the edge ij. We define the energy function of a representation $i \mapsto \mathbf{x}_i$ by

$$\mathcal{E}_w(\mathbf{x}) = \sum_{ij \in E} w_{ij} |\mathbf{x}_i - \mathbf{x}_j|^2.$$

The simple arguments above remain valid: $\mathcal{E}_w$ is strictly convex if at least one node is nailed, there is a unique optimum, and for the optimal representation every $i \in V \setminus S$ satisfies

$$\sum_{j \in N(i)} w_{ij}(\mathbf{x}_i - \mathbf{x}_j) = 0. \tag{1.6}$$

This we can rewrite as

$$\mathbf{x}_i = \frac{1}{\sum_{j \in N(i)} w_{ij}} \sum_{j \in N(i)} w_{ij} \mathbf{x}_j. \tag{1.7}$$

Thus $\mathbf{x}_i$ is no longer in the center of gravity of its neighbors, but it is still a convex combination of them with positive coefficients. In other words, it is in the relative interior of the convex hull of its neighbors.

Exercise 1.1.2 Prove that $\mathcal{E}_{\min}(w) := \min_{\mathbf{x}} \mathcal{E}_w(\mathbf{x})$ (where the minimum is taken over all representations $\mathbf{x}$ with some nodes nailed) is a concave function of w.

Exercise 1.1.3 Let $G = (V, E)$ be a connected graph, $\emptyset \neq S \subseteq V$, and $\mathbf{x}^0 : S \to \mathbb{R}^d$. Extend $\mathbf{x}^0$ to $\mathbf{x} : V \setminus S \to \mathbb{R}^d$ as follows: starting a random walk at j, let i be the (random) node where S is first hit, and let $\mathbf{x}_j$ denote the expectation of the vector $\mathbf{x}_i^0$. Prove that $\mathbf{x}$ is the same as the rubber band extension of $\mathbf{x}^0$.

1.2 Rubber bands, planarity and polytopes

1.2.1 How to draw a graph?

The rubber band method was first analyzed by Tutte [1]. In this classical paper he describes how to use "rubber bands" to draw a 3-connected planar graph with straight edges and convex faces.

Let $G = (V, E)$ be a 3-connected planar graph, and let p_0 be any country of it. Let C_0 be the cycle bounding p_0. Let us nail the nodes of C_0 to the vertices of a convex polygon P_0

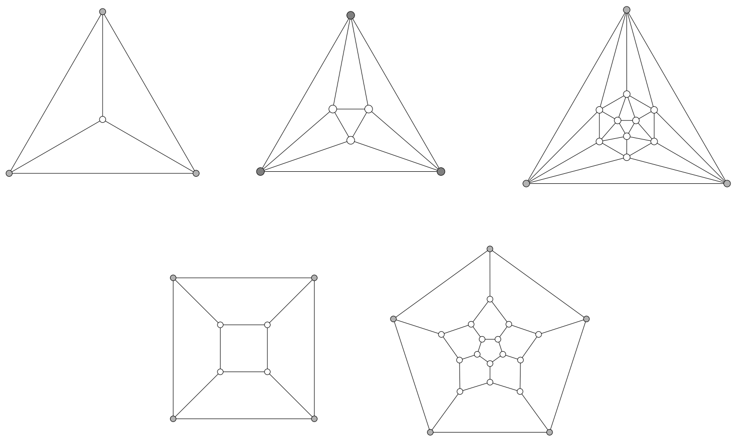

Figure 1.2: Rubber band representations of the skeletons of platonic bodies

in the plane, in the same cyclic order. Let $i \mapsto v_i$ be the rubber band representation of G in the plane extending this map. We draw the edges of G as straight line segments connecting the appropriate endpoints. Figure 1.2 shows the rubber band representation of the skeletons of the five platonic bodies.

By the above, we know that each node not on C_0 is positioned at the center of gravity of its neighbors. Tutte's main result about this embedding is the following:

Theorem 1.2.1 *If G is a simple 3-connected planar graph, then every rubber band representation of G (with the nodes of a country p_0 nailed to a convex polygon) gives an embedding of G in the plane.*

Proof. Let $G = (V, E)$, and let $\mathbf{x} : V \to \mathbb{R}^2$ be a rubber band representation of G. Let ℓ be a line intersecting the interior of the polygon P_0, and let U_0, U_1 and U_2 denote the sets of nodes of G mapped on ℓ and on the two (open) sides of ℓ, respectively.

The key to the proof is the following claim.

Claim 1. *The sets U_1 and U_2 induce connected subgraphs of G.*

Clearly the nodes of p_0 in U_1 form a (nonempty) path P_1, and similarly for U_2. We may assume that ℓ is not parallel to any edge. Let $a \in U_1 \setminus V(C_0)$, we show that it is connected to P_1 by a path in U_1. If a has a neighbor a_1 such that $\mathbf{x}_{a_1}$ is in U_1, but farther away from ℓ than $\mathbf{x}_a$, then either a_1 is nailed (and we are done), or we can find a neighbor a_2 of a_1 such that $\mathbf{x}_{a_2} \in U_1$ but farther from ℓ than $\mathbf{x}_{a_1}$, etc. This way we get a path Q in G that starts at a, and stays in U_1, and eventually must hit P_1.

Suppose that a has no neighbor a_1 as used above. Consider all nodes represented by $\mathbf{x}_a$, and the connected component H of the subgraph of G induced by them containing a. If H

contains a nailed node, then it contains a path from a to P_1, all in U_1. Else, there must be an edge connecting a node $a_1 \in V(H)$ to a node outside H (since G is connected). Since the system is in equilibrium, a_1 must have a neighbor a_2 such that $\mathbf{x}_{a_2}$ is farther away from ℓ than $\mathbf{x}_a = \mathbf{x}_{a_1}$ (here we use that no edge is parallel to ℓ). We know already that a_1 can be connected by a path in U_1 to P_1, and a can be connected to a_1 in H. This proves the claim (Figure 1.3).

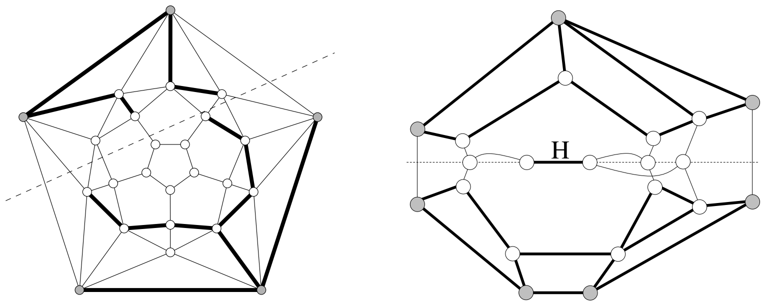

Figure 1.3: Left: every line cuts a rubber band representation into connected parts. Right: Each node on a line must have neighbors on both sides.

Next, we exclude a couple of possible degeneracies.

Claim 2. *Every $u \in U_0$ has neighbors in both U_1 and U_2.*

This is trivial if $u \in V(C_0)$, so suppose that u is a free node. If u has a neighbor in U_1, then it must also have a neighbor in U_2; follows from the fact $\mathbf{x}_u$ is the center of gravity of the points $\mathbf{x}_v$, $v \in N(u)$. So it suffices to prove that not all neighbors of u are contained in U_0.

Let T be the set of nodes $u \in U_0$ with $N(u) \subseteq U_0$, and suppose that this set is nonempty. Consider a connected component H of $G[T]$ (H may be a single node), and let S be the set of neighbors of H outside H. Since $V(H) \cup S \subseteq U_0$, it cannot contained all nodes, and hence S is a cutset. Thus $|S| \geq 3$ by 3-connectivity.

If $a \in S$, then $a \in U_0$ by the definition of S, but a has a neighbor not in U_0, and so it has neighbors in both U_1 and U_2 by the argument above (see Figure 1.3). The set $V(H)$ induces a connected graph by definition, and U_1 and U_2 induce connected subgraphs by Claim 1. So we can contract these sets to single nodes. These three nodes will be adjacent to all nodes in S. So G can be contracted to $K_{3,3}$, which is a contradiction since it is planar. This proves Claim 2.

Claim 3. *Every country has at most two nodes in U_0.*

Suppose that $a, b, c \in U_0$ are nodes of a country p. Clearly $p \neq p_0$. Let us create a new node d and connect it to a, b and c; the resulting graph G' is still planar. On the other hand, the same argument as in the proof of Claim 2 (with $V(H) = d$ and $S = \{a, b, c\}$) shows that

G' has a $K_{3,3}$ minor, which is a contradiction.

Claim 4. *Let p and q be the two countries sharing an edge ab, where $a, b \in U_0$. Then $V(p_1) \setminus \{a, b\} \subseteq U_1$ and $V(p_2) \setminus \{a, b\} \subseteq U_2$ (or the other way around).*

Suppose not, then p has a node $c \neq a, b$ and q has a node $d \neq a, b$ such that (say) $c, d \in U_1$. (Note that $c, d \notin U_0$ by Claim 3.) By Claim 1, there is a path P in U_1 connecting c and d (Figure 1.4). Claim 2 implies that both a and b have neighbors in U_2, and again Claim 1, these can be connected by a path in U_2. This yields a path P' connecting a and b whose inner nodes are in U_2. By their definition, P and P' are node-disjoint. But look at any planar embedding of G: the edge ab, together with the path P', forms a Jordan curve that does not go through c and d, but separates them, so P cannot exist.

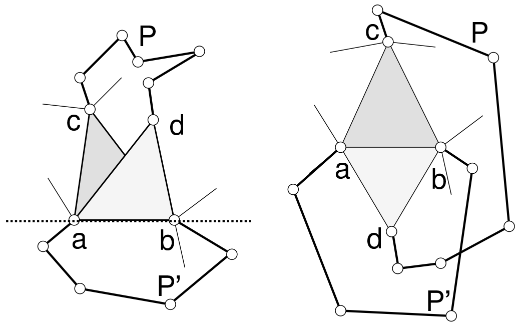

Figure 1.4: Two adjacent countries having nodes on the same side of ℓ (left), and the supposedly disjoint paths in the planar embedding (right).

Claim 5. *The boundary of every country q is mapped onto a convex polygon P_q.*

This is immediate from Claim 4, since the line of an edge of a country cannot intersect its interior.

Claim 6. *The interiors of the polygons P_q $(q \neq p_0)$ are disjoint.*

Let $\mathbf{x}$ be a point inside P_{p_0}, we want to show that it is covered by one P_q only. Clearly we may assume that $\mathbf{x}$ is not on the image of any edge. Draw a line through $\mathbf{x}$ that does not go through the image any node, and see how many times its points are covered by interiors of such polygons. As we enter P_{p_0}, this number is clearly 1. Claim 4 says that as the line crosses an edge, this number does not change. So $\mathbf{x}$ is covered exactly once.

Now the proof is essentially finished. Suppose that the images of two edges have a common point. Then two of the countries incident with them would have a common interior point, which is a contradiction except if these countries are the same, and the two edges are consecutive edges of this country. $\square$

Before going on, let's analyze this proof a little. The key step, namely Claim 1, is very similar to a basic fact concerning skeletons of convex polytopes, namely that its vertices in every (open or closed) halfspace induce a connected (or empty) subgraph. Let us call a geometric representation of a graph *section-connected*, if for every open halfspace, the subgraph induced by those nodes that are mapped into this halfspace is connected (or empty). The skeleton of a polytope, as a representation of itself, is section-connected; and so is the rubber-band representation of a planar graph. Note that the proof of Claim 1 did not make use of the planarity of G; in fact, the same proof gives:

Lemma 1.2.2 *Let G be a connected graph, and let w be a geometric representation of an induced subgraph H of G (in any dimension). If w is section-connected, then its rubber-band extension to G is section-connected as well.*

1.2.2 How to lift a graph?

An old construction of Cremona and Maxwell can be used to "lift" Tutte's rubber band representation to a Steinitz representation.

Theorem 1.2.3 *Let $G = (V, E)$ be a 3-connected planar graph, and let T be a triangular country of G. Let*

$$i \mapsto \mathbf{u}_i = \begin{pmatrix} u_{i1} \\ u_{i2} \end{pmatrix} \in \mathbb{R}^2$$

be a rubber band representation of G obtained by nailing T to any triangle in the plane. Then we can assign a number $\eta_i \in \mathbb{R}$ to each $i \in V$ such that $\eta_i = 0$ for $i \in V(T)$, $\eta_i > 0$ for $i \in V \setminus V(T)$, and the mapping

$$i \mapsto \mathbf{v}_i = \begin{pmatrix} \mathbf{u}_i \\ \eta_i \end{pmatrix} = \begin{pmatrix} u_{i1} \\ u_{i2} \\ \eta_i \end{pmatrix} \in \mathbb{R}^3$$

is a Steinitz representation of G.

Example 1.2.4 Consider the rubber band representation of a triangular prism in Figure 1.5. If this is a projection of a convex polyhedron, then the lines of the three edges pass through one point: the point of intersection of the planes of the three quadrangular faces. It is easy to see that this condition is necessary and sufficient. To see that it is satisfied by a rubber band representation, it suffices to node that the inner triangle is in equilibrium, and this implies that the lines of action of the forces acting on it must pass through one point.

Before starting with the proof, we need a little preparation to deal with edges on the boundary triangle. Recall that we can think of $\mathbf{F}_{ij} = \mathbf{u}_i - \mathbf{u}_j$ as the force with which the edge ij pulls its endpoint j. Equilibrium means that for every internal node j,

$$\sum_{i \in N(j)} \mathbf{F}_{ij} = 0. \tag{1.8}$$

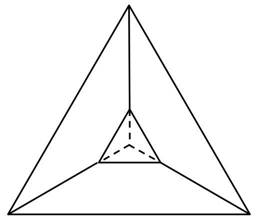

Figure 1.5: The rubber band representation of a triangular prism is the projection of a polytope.

This does not hold for the nailed nodes, but we can modify the definition of $\mathbf{F}_{ij}$ along the three boundary edges so that (1.8) will hold for all nodes (this is the only point where we use that the outer country is a triangle). This is natural by a physical argument: let us replace the outer edges by rigid bars, and remove the nails. The whole structure will remain in equilibrium, so appropriate forces must act in the edges ab, bc and ac to keep balance. To translate this to mathematics, one has to work a little; this is left to the reader as Exercise 1.2.6.

Now we are ready to prove theorem 1.2.3.

Proof. Imagine that we have found a lifting with the properties required in the theorem. Let's call the third coordinate direction "vertical". For each face F, let $\mathbf{g}_F$ be a normal vector. Since no face is parallel to a vertical line, we can normalize $\mathbf{g}_F$ so that its third coordinate is 1. Clearly for each face F, $\mathbf{g}_F$ will be an outer normal, except for $F = T$, when $\mathbf{g}_F$ is an inner normal.

Write $\mathbf{g}_F = \binom{\mathbf{h}_F}{1}$. Let ij be any edge of G, and let p and q be the two countries incident with ij. Then both $\mathbf{g}_p$ and $\mathbf{g}_q$ are orthogonal to the edge $\mathbf{v}_i\mathbf{v}_j$ of the polytope, and therefore so is their difference, and so

$$(\mathbf{h}_p - \mathbf{h}_q)^\mathsf{T}(\mathbf{u}_i - \mathbf{u}_j) = \begin{pmatrix} \mathbf{h}_p - \mathbf{h}_q \\ 0 \end{pmatrix}^\mathsf{T} \begin{pmatrix} \mathbf{u}_i - \mathbf{u}_j \\ \eta_i - \eta_j \end{pmatrix} = (\mathbf{g}_p - \mathbf{g}_q)^\mathsf{T}(\mathbf{v}_i - \mathbf{v}_j) = 0. \tag{1.9}$$

We have $\mathbf{h}_T = 0$, since the facet T is horizontal.

Using that not only $\mathbf{g}_p - \mathbf{g}_q$, but also $\mathbf{g}_p$ is orthogonal to $\mathbf{v}_i - \mathbf{v}_j$, we get that

$$\eta_i - \eta_j = \mathbf{g}_p^\mathsf{T}(\mathbf{v}_i - \mathbf{v}_j) - \mathbf{h}_p^\mathsf{T}(\mathbf{u}_i - \mathbf{u}_j) = -\mathbf{h}_p^\mathsf{T}(\mathbf{u}_i - \mathbf{u}_j). \tag{1.10}$$

This discussion allows us to explain the plan of the proof: given the Tutte representation, we first reconstruct the vectors $\mathbf{h}_F$ so that all equations (1.9) are satisfied, then using these, we reconstruct the numbers η_i so that equations (1.10) are satisfied. It will not be hard to verify then that we get a Steinitz representation.

Let R denote the counterclockwise rotation in the plane by 90°. We claim that we can replace (1.9) by the stronger equation

$$\mathbf{h}_p - \mathbf{h}_q = R\mathbf{F}_{ij} \tag{1.11}$$

and still have a solution. Starting with $\mathbf{h}_T = 0$, and moving from face to adjacent face, this equation will determine the value of $\mathbf{h}_F$ for every face. What we have to show is that we don't run into contradiction, i.e., if we get to the same face F in two different ways, then we get the same vector $\mathbf{h}_F$. This is equivalent to saying that if we walk around a closed cycle of faces, then the total change in the vector $\mathbf{h}_F$ is zero. It suffices to verify this when we move around countries incident with a single node. In this case, we have to verify that

$$\sum_{i\in N(j)} R\mathbf{F}_{ij} = 0,$$

which follows by (1.8). This proves that the vectors $\mathbf{h}_F$ are well defined.

Second, we construct numbers η_i satisfying (1.10) by a similar argument (just working on the dual graph). We set $\eta_i = 0$ if i is an external node. Equation (1.10) tells us what the value at one endpoint of an edge must be, if we have it for the other endpoint.

One complication is that (1.10) gives two conditions for each difference $\eta_i - \eta_j$, depending on which country incident with it we choose. But if p and q are the two countries incident with the edge ij, then

$$\mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) - \mathbf{h}_q^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) = (\mathbf{h}_p - \mathbf{h}_q)^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) = (R\mathbf{F}_{ij})^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) = 0,$$

since $\mathbf{F}_{ij}$ is parallel to $\mathbf{u}_i - \mathbf{u}_j$ and so $R\mathbf{F}_{ij}$ is orthogonal to it. Thus the two conditions on the difference $\eta_i - \eta_j$ are the same.

As before, equation (1.10) determines the values η_i, starting with $\eta_a = 0$. To prove that it does not lead to a contradiction, it suffices to prove that the sum of changes is 0 if we walk around a face p. In other words, if $\overrightarrow{C}$ is the cycle bounding a face p (oriented, say, clockwise), then

$$\sum_{\overrightarrow{ij}\in E(\overrightarrow{C})} \mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) = 0,$$

which is clear. It is also clear that $\eta_b = \eta_c = 0$.

Now define $\mathbf{v}_i = \binom{\mathbf{u}_i}{\eta_i}$ for every node i, and $\mathbf{g}_p = \binom{\mathbf{h}_p}{1}$ for every face p. It remains to prove that $i \mapsto \mathbf{v}_i$ maps the nodes of G onto the vertices of a convex polytope, so that edges go to edges and countries go to faces. We start with observing that if p is a face and ij is an edge of p, then

$$\mathbf{g}_p^{\mathsf{T}}\mathbf{v}_i - \mathbf{g}_p^{\mathsf{T}}\mathbf{v}_j = \mathbf{h}_p^{\mathsf{T}}(\mathbf{u}_i - \mathbf{u}_j) + (\eta_i - \eta_j) = 0,$$

and hence there is a scalar α_p so that all nodes of p are mapped onto the hyperplane $\mathbf{g}_p^{\mathsf{T}}\mathbf{x} = \alpha_p$. We know that the image of p under $i \mapsto \mathbf{u}_i$ is a convex polygon, and so the same follows for the map $i \mapsto \mathbf{v}_i$.

To conclude, it suffices to prove that if ij is any edge, then the two convex polygons obtained as images of countries incident with ij "bend" in the right way; more exactly, let p

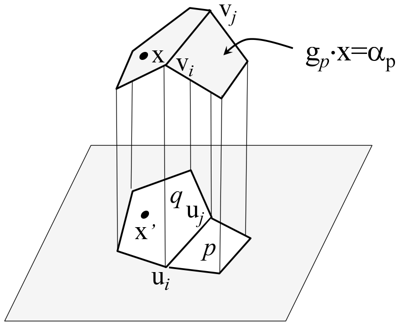

Figure 1.6: Lifting a rubber band representation to a polytope.

and q be the two countries incident with ij, and let Q_p and Q_q be two corresponding convex polygons (see Figure 1.6). We claim that Q_p lies on the same side of the plane $\mathbf{g}_p^{\mathsf{T}}\mathbf{x} = \alpha_p$ as the bottom face. Let $\mathbf{x}$ be any point of the polygon Q_q not on the edge $\mathbf{v}_i\mathbf{v}_j$. We want to show that $\mathbf{g}_p^{\mathsf{T}}\mathbf{x} < \alpha_p$. Indeed,

$$\mathbf{g}_p^{\mathsf{T}}\mathbf{x} - \alpha_p = \mathbf{g}_p^{\mathsf{T}}\mathbf{x} - \mathbf{g}_p^{\mathsf{T}}\mathbf{v}_i = \mathbf{g}_p^{\mathsf{T}}(\mathbf{x} - \mathbf{v}_i) = (\mathbf{g}_p - \mathbf{g}_q)^{\mathsf{T}}(\mathbf{x} - \mathbf{v}_i)$$

(since both $\mathbf{x}$ and $\mathbf{v}_i$ lie on the plane $\mathbf{g}_q^{\mathsf{T}}\mathbf{x} = \alpha_q$),

$$= \begin{pmatrix} \mathbf{h}_p - \mathbf{h}_q \\ 0 \end{pmatrix}^{\mathsf{T}} (\mathbf{x} - \mathbf{v}_i) = (\mathbf{h}_p - \mathbf{h}_q)^{\mathsf{T}}(\mathbf{x}' - \mathbf{u}_i)$$

(where $\mathbf{x}'$ is the projection of $\mathbf{x}$ onto the first two coordinates)

$$= (R\mathbf{F}_{ij})^{\mathsf{T}}(\mathbf{x}' - \mathbf{u}_i) < 0$$

(since $\mathbf{x}'$ lies to the right from the edge $\mathbf{u}_i\mathbf{u}_j$). This completes the proof. $\square$

Theorem 1.2.3 proves Steinitz's theorem in the case when the graph has a triangular face. We are also home if the dual graph has a triangular face; then we can represent the dual graph as the skeleton of a 3-polytope, choose the origin in the interior of this polytope, and consider its polar; this will represent the original graph.

So the proof of Steinitz's theorem is complete, if we prove the following simple fact:

Lemma 1.2.5 *Let G be a 3-connected simple planar graph. Then either G or its dual has a triangular face.*

Proof. If G^* has no triangular face, then every node in G has degree at least 4, and so

$$|E(G)| \geq 2|V(G)|.$$

If G has no triangular face, then similarly

$$|E(G^*)| \geq 2|V(G^*)|.$$

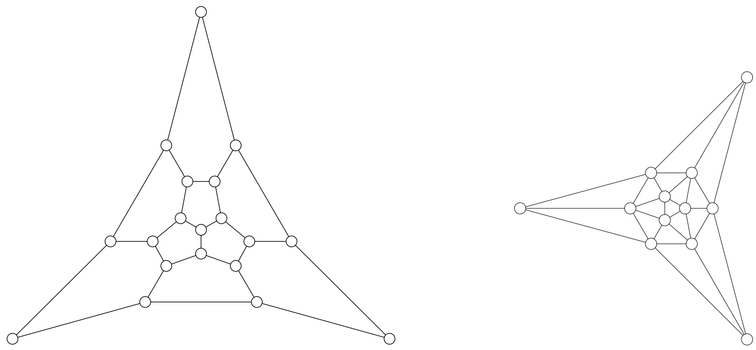

Figure 1.7: Rubber band representation of a dodecahedron with one node deleted, and of an icosahedron with the edges of a triangle deleted. Corresponding edges are parallel and have the same length.

Adding up these two inequalities and using that $|E(G)| = |E(G^*)|$ and $|V(G)|+|V(G^*)| = |E(G)|+2$ by Euler's theorem, we get

$$2|E(G)| \geq 2|V(G)|+2|V(G^*)| = 2|E(G)|+4,$$

a contradiction. □

Exercise 1.2.6 Let $\mathbf{u}$ be a rubber band representation of a planar map G in the plane with the nodes of a country T nailed to a convex polygon. Define $\mathbf{F}_{ij} = \mathbf{u}_i - \mathbf{u}_j$ for all edges in $E \setminus E(T)$. (a) If T is a triangle, then we can define $\mathbf{F}_{ij} \in \mathbb{R}^2$ for $ij \in E(T)$ so that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$, $\mathbf{F}_{ij}$ is parallel to $\mathbf{u}_j - \mathbf{u}_i$, and $\sum_{i \in N(j)} \mathbf{F}_{ij} = 0$ for every node i. (b) Show by an example that (a) does not remain true if we drop the condition that T is a triangle.

Exercise 1.2.7 Prove that every Schlegel diagram with respect to a face F can be obtained as a rubber band representation of the skeleton with the vertices of the face nailed (the strengths of the rubber bands must be chosen appropriately).

Exercise 1.2.8 Let G be a 3-connected planar graph with a triangular country $p = \overline{abc}$. Let q, r, s be the countries adjacent to p. Let G^* be the dual graph. Consider a rubber band representation $\mathbf{x} : V(G) \to \mathbb{R}^2$ of G with a, b, c nailed down (both with unit rubber band strengths). Prove that the segments representing the edges can be translated so that they form a rubber band representation of $G^* - p$ with q, r, s nailed down (Figure 1.7).

Exercise 1.2.9 A *convex representation* of a graph $G = (V, E)$ (in dimension d, with boundary $S \subseteq V$) is an mapping of $V \to \mathbb{R}^d$ such that every node in $V \setminus S$ is in the relative interior of the convex hull of its neighbors.(a) The rubber band representation extending any map from $S \subseteq V$ to $\mathbb{R}^d$ is convex with boundary S. (b) Not every convex representation is constructible this way.

Bibliography

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