Chapter 4

Square tilings

We can also represent planar graphs by squares, rather then circles, in the plane (with some mild restrictions). There are in fact two quite different ways of doing this: the squares can correspond to the edges (a classic result of Brooks, Smith, Stone and Tutte), or the squares can correspond to the nodes (a more recent result of Schramm).

4.1 Electric current through a rectangle

A beautiful connection between square tilings and harmonic functions was described in the classic paper of Brooks, Smith, Stone and Tutte [1]. They considered tilings of squares by smaller squares, and used a physical model of current flows to show that such tilings can be obtained from any connected planar graph. Their ultimate goal was to construct tilings of a square with squares whose edge-lengths are all different; this will not be our concern; we'll allow squares that are equal and also the domain being tiled can be a rectangle, not necessarily a square.

Consider tiling $\mathcal{T}$ of a rectangle R with a finite number of squares, whose sides are parallel to the coordinate axes. We can associate a planar map with this tiling as follows. Represent any maximal horizontal segment composed of edges of the squares by a single node (say, positioned at the midpoint of the segment). Each square "connects" two horizontal segments, and we can represent it by an edge connecting the two corresponding nodes, directed top-down. We get a directed graph $G_{\mathcal{T}}$ (Figure 4.1), with a single source s (representing the upper edge of the rectangle) and a single sink t (representing the upper edge). It is easy to see that graph $G_{\mathcal{T}}$ is planar.

A little attention must be paid to points where four squares meet. Suppose that squares A, B, C, D share a corner p, where A is the upper left, and B, C, D follow clockwise. In this case, we may consider the lower edges of A and B to belong to a single horizontal segment, or to belong to different horizontal segments. In the latter case, we may or may not imagine that there is an infinitesimally small square sitting at p. What this means is that we have to

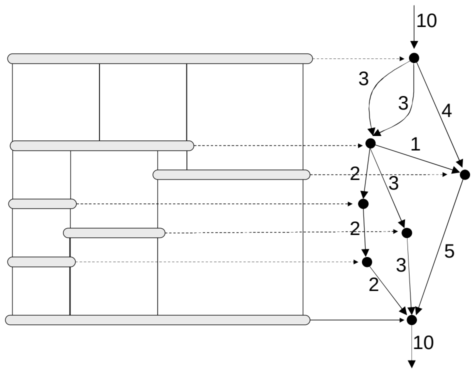

Figure 4.1: The Brooks–Smith–Stone–Tutte construction

declare if the four edges of $G_{\mathcal{T}}$ corresponding to A, B, C and D form two pairs of parallel edges, an empty quadrilateral, or a quadrilateral with a horizontal diagonal. We can orient this horizontal edge arbitrarily (Figure 4.2).

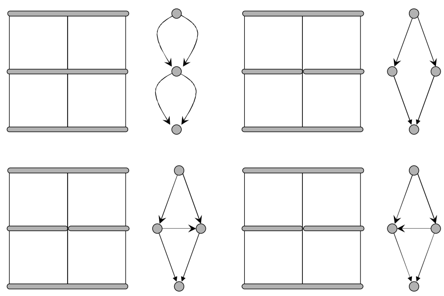

Figure 4.2: Possible declarations about four squares meeting at a point

If we assign the edge length of each square to the corresponding edge, we get a flow f from s to t: If a node v represents a segment I, then the total flow into v is the sum of edge length of squares attached to I from the top, while the total flow out of v is the sum of edge length of squares attached to I from the bottom. Both of these sums are equal to the length of I.

Let $h(v)$ denote the distance of node v from the upper edge of R. Since the edge-length of a square is also the difference between the y-coordinates of its upper and lower edges, the function h is harmonic:

$$h(i) = \frac{1}{d_i} \sum_{j \in N(i)} h(j)$$

for every node different from s and t (Figure 4.1).

It is not hard to see that this construction can be reversed:

Theorem 4.1.1 *For every connected planar map G with two specified nodes s and t on the unbounded face, there is a unique tiling $\mathcal{T}$ of a rectangle such that $G \cong G_{\mathcal{T}}$.*

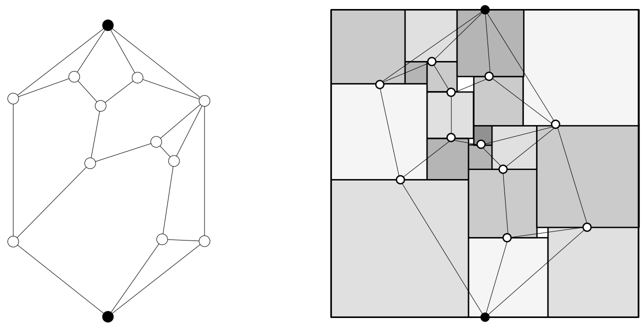

Figure 4.3: A planar graph and the square tiling generated from it.

4.2 Tangency graphs of square tilings

Let R be a rectangle in the plane, and consider a tiling of R by squares. Let us add four further squares attached to each edge of R from the outside, sharing the edge with R. We want to look at the tangency graph of this family of squares, which we call shortly *the tangency graph of the tiling of R.*

The tangency graph of the squares may not be planar. Let us try to draw the tangency graph in the plane by representing each square by its center and connecting the centers of touching squares by a straight line segment. Similarly as in the preceding section, we get into trouble when four squares share a vertex, in which case two edges will cross at this vertex. In this case we specify arbitrarily one diametrically opposite pair as "infinitesimally overlapping", and connect the centers of these two but not the other two centers. We call this a *resolved tangency graph.*

Every resolved tangency graph is planar, and it is easy to see that the unbounded country is a quadrilateral, and all other countries are triangles; briefly, it is a triangulation of a quadrilateral (Figure 4.4). It is easy to see that this graph does not contain a separating 3-cycle.

Under somewhat stronger conditions, this fact has a converse, due to Schramm [2].

Theorem 4.2.1 *Every planar map in which the unbounded country is a quadrilateral, all other countries are triangles, and is not separated by a 3-cycle or 4-cycle, can be represented as a resolved tangency graph of a square tiling of a rectangle.*

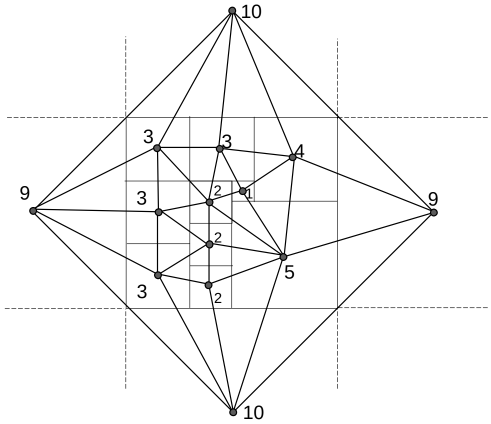

Figure 4.4: The resolved tangency graph of a tiling of a rectangle by squares. The numbers indicate the edge length of the squares

Schramm proves a more general theorem, in which separating cycles are allow; the prize to pay is that he must allow degenerate squares with edge-length 0. It is easy to see that a separating triangle forces everything inside it to degenerate in this sense, and so we don't loose anything by excluding these. Separating 4-cycles may or may not force degeneracy, and it does not seem easy to tell when they do.

Before proving this theorem, we need a lemma from combinatorial optimization. Let G be a planar triangulation of a quadrilateral *abcd*. We assign a real variable x_i to each node $i \neq a, b, c, d$ (this will eventually mean the side length of the square representing i, but at the moment, it is just a variable). Let $V'(P)$ denote the set of internal nodes of a path P, and consider the following conditions:

$$x_i \geq 0 \qquad \text{for all nodes } i, \tag{4.1}$$

$$\sum_{i \in V'(P)} x_i \geq 1 \qquad \text{for all } a\text{–}c \text{ paths } P. \tag{4.2}$$

Let $K \subseteq \mathbb{R}^V$ denote the solution set of these inequalities. It is clear that K is an ascending polyhedron.

Lemma 4.2.2 *The vertices of K are indicator vectors of sets $V'(Q)$, where Q is a b–d paths. The blocker of K is defined by the inequalities*

$$x_i \geq 0 \qquad \textit{for all nodes } i, \tag{4.3}$$

$$\sum_{i \in V'(Q)} x_i \geq 1 \qquad \textit{for all } b\text{–}d \textit{ paths } Q. \tag{4.4}$$

The vertices of K^{bl} are indicator vectors of sets $V'(P)$, where P is an a–c path.

These facts can be derived from the Max-Flow-Min-Cut Theorem.

Proof of Theorem 4.2.1. Consider the solution $\overline{x}$ of (4.1)–(4.2) minimizing the objective function $\sum_i x_i^2$, and let R^2 be the minimum value. By Duality Theorem for blocking polyhedra (see the Background Material), $\overline{y} = (1/R^2)\overline{x}$ minimizes the same objective function over $K^{\rm bl}$. Let us rescale these vectors to get $z = \frac{1}{R}\overline{x} = R\overline{y}$.

It will be convenient to define $z_a = z_c = R$ and $z_b = z_d = 1/R$. then we have

$$\sum_{i\in V'(P)} z_i \geq \frac{1}{R} \qquad \text{for all } a\text{–}c \text{ paths } P \tag{4.5}$$

$$\sum_{i\in V'(Q)} z_i \geq R \qquad \text{for all } b\text{–}d \text{ paths } Q \tag{4.6}$$

$$\sum_{i\in V\setminus\{a,b,c,d\}} z_i^2 = 1. \tag{4.7}$$

For each edge ij, we define its z-*length* as $\frac{1}{2}z_i + \frac{1}{2}z_j$. This defines the z-length of any path as the sum of z-lengths of its edges, and the z-*distance* of $d(u,v)$ of two nodes u and v, as the minimum z-length of a path connecting them. It is clear that for adjacent nodes i and j, we have $d(i,j) = \frac{1}{2}z_i + \frac{1}{2}z_j$. Using the trivial observation that in the optimum solution of (4.1)–(4.2) at least one constraint must be satisfied with equality, we can express the conditions on z as $d(a,c) = d(b,d) = R + \frac{1}{R}$.

We can tell now what will be the squares representing G: every node i will be represented by a square S_i with center $\mathbf{p}_i = (d(a,i), d(b,i))$ and side z_i. To prove that this construction works will take substantially more work.

Let us call an a–c path *tight*, if equality holds in (4.5), and let us define tight b–d paths analogously. The paths abc, adc, bad and bcd are tight. It follows from the optimality of $\overline{x}$ and $\overline{y}$ that every node i with $z_i > 0$ is contained in a tight a–c path as well as in a tight b–d path.

Tight paths have the following crucial property. Let P and P' be two tight a–c paths, and suppose that they have an internal node w in common. Let $P[a,w]$ denote the subpath of P between a and w, then $S = P[a,w] \cup P'[w,c]$ and $T = P'[a,w] \cup P[w,c]$ are connected subgraphs containing a and c, and hence they contain a–c paths S_0 and T_0. But then

$$\frac{2}{R} = z(V'(P)) + z(V'(P')) \geq z(V'(S_0)) + z(V'(T_0)) \geq \frac{2}{R}.$$

It follows that S_0 and T_0 must be tight paths themselves. We say that these paths arise from P and P' by *swapping at* w.

We know that $\overline{y}$ is in the polyhedron defined by (4.1)–(4.2), and so it can be written as a convex combination of vertices, which are indicator vectors of sets $V'(P)$, where P is an a–c path. Let $\mathbb{1}_P$ denote the indicator vector of $V(P)$ (the endpoints included), then the corresponding convex combination of vectors $\mathbb{1}_P$ gives 1 at the nodes a and c. Hence z, restricted to $V\setminus\{b,d\}$, can be written as

$$z = \sum_{P\in\mathcal{P}} \lambda_P \mathbb{1}_P, \tag{4.8}$$

where $\mathcal{P}$ is a set of a–c paths, $\lambda_P > 0$ and $\sum_P \lambda_P = R$. Similarly, we have a decomposition

$$z = \sum_{Q\in\mathcal{Q}} \mu_Q \mathbb{1}_Q, \tag{4.9}$$

where the $\mathcal{Q}$ is a set of b–d paths, $\mu_j > 0$ and $\sum_j \mu_j = 1/R$. Trivially a node v has $z_v > 0$ if and only if it is contained in one of the paths in $\mathcal{P}$ (equivalently, in one of the paths in $\mathcal{Q}$).

Claim 1 $|V(P)\cap V(Q)| = 1$ *for all* $P \in \mathcal{P}, Q \in \mathcal{Q}$.

It is trivial from the topology of G that $|V(P)\cap V(Q)| \geq 1$. On the other hand, (4.7) implies that

$$\begin{aligned} 1 = \sum_{i\in V\setminus\{a,b,c,d\}} z_i^2 &= \Big(\sum_P \lambda_P \mathbb{1}_P\Big)^{\mathsf{T}} \Big(\sum_Q \mu_Q \mathbb{1}_Q\Big) = \sum_{P,Q} \lambda_P \mu_Q |V(P)\cap V(Q)| \\ &\geq \sum_{P,Q} \lambda_P \mu_Q = \Big(\sum_P \lambda_P\Big)\Big(\sum_Q \mu_Q\Big) = 1. \end{aligned}$$

We must have equality here, which proves the Claim.

Claim 2 *All paths in* $\mathcal{P}$ *and* $\mathcal{Q}$ *are tight.*

Indeed, if (say) $Q \in \mathcal{Q}$, then

$$\sum_{i\in V'(Q)} z_i = z^{\mathsf{T}} \mathbb{1}_Q = \sum_{P\in\mathcal{P}} \lambda_P \mathbb{1}_P^{\mathsf{T}} \mathbb{1}_Q = \sum_{P\in\mathcal{P}} \lambda_P |V(P)\cap V(Q)| = \sum_{P\in\mathcal{P}} \lambda_P = R.$$

One consequence of this claim is that the paths in $\mathcal{P}$ and $\mathcal{Q}$ are chordless, since if (say) $P \in \mathcal{P}$ had a cord uv, then bypassing the part of P between u and v would decrease its length, which would contradict (4.5).

Another way of say Claim 2 is that all paths in $\mathcal{P}$ are shortest a–c paths (with respect to the z-length). This implies that all their subpaths are shortest as well. Then for any $u \in V'(P)$, this last observation implies that

$$d(a,u) = \frac{z_a}{2} + \frac{z_u}{2} + \sum_{i\in V'(P[a,u])} z_i. \tag{4.10}$$

Let us fix a node $u \neq a,b,c,d$. Every path in $\mathcal{P}$ goes either "left" of u (i.e., separates u from b), or through u, or "right" of u. Let $\mathcal{P}_u^-$, $\mathcal{P}_u$ and $\mathcal{P}_u^+$ denote these three sets. Similarly, we can partition $\mathcal{Q} = \mathcal{Q}_u^- \cup \mathcal{Q}_u \mathcal{Q}_u^+$ according to whether a path in $\mathcal{Q}$ goes "above" u (separating u from a), or through u, or "below" u. If $P \in \mathcal{Q}$ is any path through u, then it is easy to tell which paths in $\mathcal{Q}$ go above u: exactly those, whose unique common node with P is between a and u. Hence

$$\begin{aligned} d(a,u) &= \frac{z_a}{2} + \frac{z_u}{2} + \sum_{i \in V'(P[a,u])} z_i = \frac{z_a}{2} + \frac{z_u}{2} + \sum_{i \in V'(P[a,u])} \mu(\mathcal{Q}_i) \\ &= \frac{R}{2} + \frac{1}{2}\mu(\mathcal{Q}_u) + \mu(\mathcal{Q}_u^-). \end{aligned} \tag{4.11}$$

Claim 3 *Every node i has $z_i > 0$.*

Suppose that there are nodes with $z_i = 0$, and let H be a connected component of the subgraph induced by these nodes. Since no path in $\mathcal{P}$ goes through any node in H, every path in $\mathcal{P}$ goes either left of all nodes in H or right of all nodes in H. Every neighbor v of H has $z_v > 0$, and hence it is contained both on a path in $\mathcal{P}$ and a path in $\mathcal{Q}$. By our assumption that there are no separating 3- and 4-cycles, H has at least 5 neighbors, so there must be two neighbors v and v' of H that are both contained (say) in paths in $\mathcal{P}$ to the left of H and in paths in Q above H. Let $P \in \mathcal{P}$ go through v and left of H, and let $Q \in \mathcal{Q}$ go through v and above H. Let P' and Q' be defined analogously.

The paths P and Q intersect at v and at no other node, by Claim 1. Clearly H must be in the (open) region T bounded by $P[v,c] \cup Q[v,d] \cup \{cd\}$, and since v' is a neighbor of H, it must be in T or on its boundary. If $v' \in V'(P[v,c]$, then Q' goes below v, hence Q' goes below the neighbor of v in H, which contradicts its definition. We get a similar contradiction if $v' \in V'(Q(v,d))$. Finally if $v' \in T$, then both P' and Q' must enter T (when starting at a and b, respectively). Since P' is to the left of v, its unique intersection with Q must be on $Q[a,v]$, and hence P' must enter R crossing $P[v,c]$. Similarly Q' must enter T crossing $Q[v,d]$. But then P' and Q' must have an intersection point before entering R, which together with v' violates Claim 1. This completes the proof of Claim 3.

Claim 4 *Let $ij \in E(G)$. Then one of the following possibilities hold:*

(i) *ij lies on one of the paths in $\mathcal{P}$. Then $|d(a,i) - d(a,j)| = d(i,j)$, but $|d(b,i) - d(b,j)| < d(i,j)$.*

(ii) *ij lies on one of the paths in $\mathcal{Q}$. Then $|d(b,i) - d(b,j)| = d(i,j)$, but $|d(a,i) - d(a,j)| < d(i,j)$.*

(iii) *No path in $\mathcal{P} \cup \mathcal{Q}$ goes through ij. Then $|d(b,i) - d(b,j)| = |d(a,i) - d(a,j)| = d(i,j)$.*

Claim 1 implies that no edge can be contained in a $\mathcal{P}$-path as well as in a $\mathcal{Q}$-path. If $ij \in E(P)$ for some $P \in \mathcal{P}$, then $|d(a,i) - d(a,j)| = d(i,j)$ by elementary properties of shortest paths. Every path in $\mathcal{P}$ that goes left of i goes either left of j or through it, and vice versa. Hence by (4.11),

$$\begin{aligned}
d(b,i)-d(b,j) &= \frac{1}{2}\lambda(\mathcal{P}_i)+\lambda(\mathcal{P}_i^-)-\frac{1}{2}\lambda(\mathcal{P}_j)-\lambda(\mathcal{P}_j^-)\\
&= \frac{1}{2}\lambda(\mathcal{P}_j)+\frac{1}{2}\lambda(\mathcal{P}_i)-\lambda(\mathcal{P}_i\cap\mathcal{P}_j^-)-\lambda(\mathcal{P}_j\setminus\mathcal{P}_i^-)\\
&\le \frac{1}{2}\lambda(\mathcal{P}_j)+\frac{1}{2}\lambda(\mathcal{P}_i) = d(i,j),
\end{aligned}$$

and if equality holds then $\mathcal{P}_i^- \supseteq \mathcal{P}_j$, but this would contradict the assumption that $\mathcal{P}_i\cap\mathcal{P}_j \neq \emptyset$. This proves (i). The proof of (ii) is similar.

Suppose that no path of $\mathcal{P}$ or $\mathcal{Q}$ contains the edge ij. Let $P\in\mathcal{P}_i$, and suppose that P goes left of j. We claim that every path in $\mathcal{P}_j$ goes right of i. Suppose that there is a path $P'\in\mathcal{P}_j\cap\mathcal{P}_i^-$. Then clearly P and P' must intersect at a node w such that $i\in P[a,w]$ and $j\in P'[w,c]$ (perhaps with the roles of a and c interchanged). Then swapping at w we get a tight a–c path such that ij is a chord of it, which is impossible.

So it follows that $\mathcal{P}_i^-\cup\mathcal{P}_i=\mathcal{P}_j^-$. Hence

$$d(a,j)-d(a,i)=\frac{1}{2}\lambda(\mathcal{P}_j)+\lambda(\mathcal{P}_j^-)-\frac{1}{2}\lambda(\mathcal{P}_i)-\lambda(\mathcal{P}_i^-)=\frac{1}{2}\lambda(\mathcal{P}_i)+\frac{1}{2}\lambda(\mathcal{P}_j)=d(i,j).$$

The other equation follows similarly.

We need a somewhat similar assertion concerning non-adjacent pairs.

Claim 5 *Let $i,j\in V$, $ij\notin E(G)$. Then either $|d(a,i)-d(a,j)|>\frac{1}{2}z_i+\frac{1}{2}z_j$, or $|d(b,i)-d(b,j)|>\frac{1}{2}z_i+\frac{1}{2}z_j$, or $|d(a,i)-d(a,j)|=|d(b,i)-d(b,j)|=\frac{1}{2}z_i+\frac{1}{2}z_j$.*

If there is a tight a–c path going through i and j, then

$$|d(a,i)-d(a,j)|=d(i,j)>\frac{1}{2}z_i+\frac{1}{2}z_j.$$

So we may assume that there is no tight a–c path, and similarly no tight b–d path, through i and j. Similarly as in the proof of Claim 4(iii), we get that $\mathcal{P}_i^-\cup\mathcal{P}_i=\mathcal{P}_j^-$ (or the other way around), which implies that $|d(a,j)-d(a,i)|=\frac{1}{2}z_i+\frac{1}{2}z_j$. it follows similarly that $|d(b,j)-d(b,i)|=\frac{1}{2}z_i+\frac{1}{2}z_j$

Now it is quite easy to verify that the squares S_i defined at the beginning of the proof give a representation of G. By Claims 4 and 5, we cannot have simultaneously $|d(a,i)-d(a,j)|<\frac{1}{2}z_i+\frac{1}{2}z_j$ and $|d(b,i)-d(b,j)|<\frac{1}{2}z_i+\frac{1}{2}z_j$ for any two nodes i and j, which means that the squares S_i $(i\in V)$ have no interior point in common. The three alternatives (i)–(iii) in Claim 4 imply that if $ij\in E$, then S_i and S_j have a boundary point in common. Thus we can consider G as drawn in the plane so that node i is at the center $\mathbf{p}_i$ of the square S_i, and every edge ij is a straight segment connecting $\mathbf{p}_i$ and $\mathbf{p}_j$.

This gives a planar embedding of G. Indeed, we know from Claim 4 that every edge ij is covered by the squares S_i and S_j. This implies that edges do not cross, except possibly in the degenerate case when four squares S_i, S_j, S_k, S_l share a vertex (in this clockwise order around the vertex, starting at the Northwest), and $ik, jl \in E$. Since the squares S_i and S_j are touching along their vertical edges, we have $|d(a,i) - d(a,j)| < \frac{1}{2}z_i + \frac{1}{2}z_j$ and $d(b,j) - d(b,i) = \frac{1}{2}z_i + \frac{1}{2}z_j$. The inequality implies that there is a path in $Q \in \mathcal{Q}$ through i and j; the equation implies that i and j are consecutive points on this path. Hence $ij \in E$. Similarly, $jk, kl, li \in E$, and hence i, j, k, l form a complete 4-graph. But this is impossible in a triangulation of a quadrilateral that has no separating triangles.

Next, we argue that the squares S_i $(i \in V \setminus \{a,b,c,d\})$ tile the rectangle $\mathfrak{R}$. The inequality $d(a,i) \geq \frac{1}{2}z_a + \frac{1}{2}z_i$, along with three analogous inequalities, implies that $S_i \subseteq \mathfrak{R}$ for every $(i \in V \setminus \{a,b,c,d\})$. On the other hand, every point of $\mathfrak{R}$ is contained in a finite country F of G, which is a triangle $\overline{uvw}$. The squares S_u, S_v and S_w are centered at the vertices, and each edge is covered by the two squares centered at its endpoints. Elementary geometry implies that these three squares cover the whole triangle.

Finally, we show that an appropriately resolved tangency graph of the squares S_i is equal to G. By the above, it contains G (where for edges of type (iii), the 4-corner is resolved so as to get the edge of G). Since G is a triangulation, the containment cannot be proper, so G is the whole tangency graph. $\square$

Exercise 4.2.3 Prove that if G is a resolved tangency graph of a square tiling of a rectangle, then every triangle in G is a face.

Exercise 4.2.4 Construct a resolved tangency graph of a square tiling of a rectangle, which contains a quadrilateral with a further node in its interior.

Bibliography

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