SINGULARITY FORMATION IN NONLINEAR DISPERSIVE EQUATIONS

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This series of lectures will be devoted to the description of the singularity formation for some nonlinear dispersive models. Most of our attention will be devoted to the focusing nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + u|u|^{p-1} = 0, \quad x \in \mathbb{R}^N, \quad u(t,x) \in \mathbb{C}$$

which is a canonical model in various areas of physics including in particular nonlinear optics and plasma physics. While the existence of blow up solutions for this kind of probem has been known since the 60's in relation to a simple convexity argument, it is only since the mid 90's that the first explicit examples of blow up regimes have been described. The aim of these lectures is to give an overview of the progress made for the past ten years on the description of blow up and the related concentration of the nonlinear wave. While the Schrödinger model will be our waveguide, other important models including in particular nonlinear wave problems (like wave maps) will be adressed.

The lectures will only require basic knowledge in PDE's and functional analysis.

The plan of the lectures will roughly be as follows. Some references are suggested for the reader's convenience but this is very far from an exhaustive list on the subject.

Part I: Variational techniques: [7], [3], [2], [24], [6], [8]

(i) Subcritical regime:
- Variational characterization of the ground state
- The concentration compactness lemma
- The Cazenave-Lions' proof of orbital stability.
- Perspective: kinetic models and orbital stability of galactic clusters

(ii) L^2 critical case:
- The L^2 concentration
- Orbital stability in the focusing regime
- The direct approach: structure of the linearized operator and modulation theory
- Critical mass blow up: Merle's theorem and classification of the solitary wave dynamics

Part II: The L^2 critical blow up: [1], [16], [9], [10], [11], [12], [13], [17], [18]

(i) The log-log analysis:
- Formal approach and almost self similar solutions

- Morawetz estimate: a nonlinear estimate
- Rigidity estimates
- Stability of the log-log blow up
- Sharp derivation of the blow up speed: the outgoing flux computation
- Classification of the solitary wave dynamics: the case of small super critical mass.

(ii) More blow up solutions:

- The Bourgain Wang solutions
- Merle's classification theorem: a dynamical approach
- Stability versus instability

Part III: An introduction to the super critical case: [23], [14], [15]

- Nonlinear trapping of the self similar regime
- Refined concentration estimates and blow up of the critical norm

Part IV: Energy critical wave equations: [4], [21], [22], [20], [5], [19]

(i) Global existence:

- Power nonlinearity and wave map problems
- Variational structure and harmonic maps
- The concentration of energy
- Criterion for global existence

(ii) Singularity formation:

- Blow up with arbitrary speeds.
- Stable blow up for corotational wave maps.
- The mixed energy/Morawetz Lyapounov functional.
- The flux computation and derivation of the blow up speed.

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